Q 1

Let $F \subset S$, (S metric space equipped with the metric d)

Show $\boxed{x \longmapsto d(x, F)}$ is Lipschitz where

$f(x)$ $\qquad$ $\underbrace{d(x,F)}_{f(x)} = \inf \{ d(x,y) : y \in F \}$.

Remark $\quad$ set.

(S, d) $\qquad$ $d : S \times S \longrightarrow [0,\infty)$ function

- $d(x,y) = d(y,x)$ $\qquad$ $|x-y| = |y-x|$
- $d(x,y) \geq 0$, $d(x,y)=0 \Leftrightarrow x=y$
- $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x,y,z \in S$ $\qquad$ $|x-y| \leq |x-z| + |z-y|$

Example $\quad (\mathbb{R}^n, d)$ $\qquad d(x,y) = |x-y| = \sqrt{\sum (x_i - y_i)^2}$

全集 $\quad$ 距离

$(\Omega, \mathcal{F}, \mu)$ 测度空间

全集 $\quad$ σ-代数 $\quad$ 测度 $\qquad \mu : \mathcal{F} \rightarrow [0,\infty]$.

Lipschitz

set

given a function $f : S \longrightarrow \mathbb{R}$

f is Lipschitz if $\exists L > 0$ s.t.

$$|f(x) - f(y)| \leq L \cdot d(x,y)$$

$$|f(x) - f(y)| \leq L|x-y| \quad \leftarrow \mathbb{R}^n$$

Proof: $f(x) = d(x,F) = \inf \{ d(x,y) : y \in F \}$. ← 给定

$\boxed{|f(x) - f(y)| \leq d(x,y)}$ 目的

只需证明 $f(x)-f(y)\leqslant d(x,y) \quad \forall\, x,y\in S$.

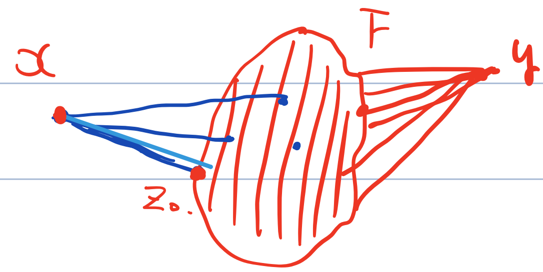
$f(y)-f(x)\leqslant d(y,x)=d(x,y)$

$-f(y)+f(x)\geqslant -d(x,y)$

$\Longrightarrow \quad -d(x,y)\leqslant f(x)-f(y)\leqslant d(x,y)$

- $f(x)=d(x,F)=\inf\{d(x,y):\ y\in F\}$. $\quad S=\mathbb{R}^2$

套inf定义 $\Big\Downarrow$

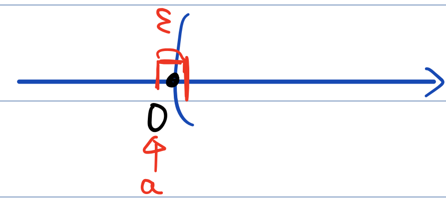


- $\forall\,\varepsilon>0,\ \exists\ \boxed{z\in F}$ s.t.

$f(x)=d(x,F)>d(x,z)-\varepsilon$

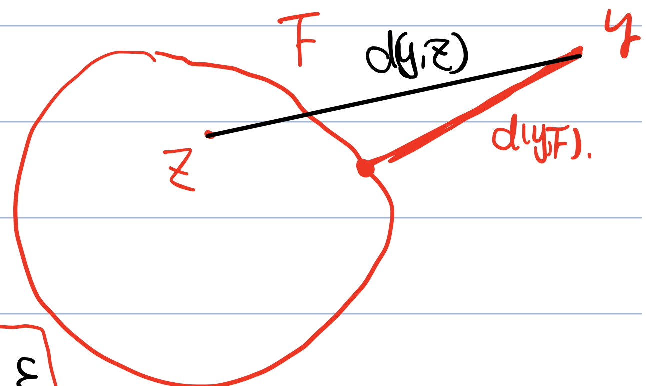
$A=(0,\infty)$

A set $\qquad \inf A=a$



- $\forall\, x\in A \quad a\leqslant x$
- $\forall\,\varepsilon>0,\ \exists\, x_0\in A$ 使得 $\quad x-\varepsilon<a$

- $d(y,F)=\inf\{d(y,w):\ w\in F\}$
$\leqslant d(y,z)$
$\leqslant d(y,x)+d(x,z)$
$\leqslant d(y,x)+d(x,F)+\varepsilon$.



$$\boxed{d(y,F)-d(x,F)\leqslant d(x,y)+\varepsilon}$$

Let $\varepsilon\to 0^+$, we have

$f(y)-f(x)=d(y,F)-d(x,F)\leqslant d(x,y)$ $\qquad\square$

$\inf\{f(a):\ a\in F\}$. $\quad x$ 给定

只要 $z \in \{ f(a) : a \in F \}$

都有 $\inf \{ f(a) : a \in F \} \leq f(z)$

Q 2.

$\{r_i\}_{i=1}^{\infty} = \mathbb{Q}$ rational. Define $F: \mathbb{R} \to \mathbb{R}$.

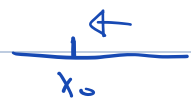
$$F(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbb{1}_{(-\infty, x]}(r_i)$$ 示性函数

Prove that F is right-continuous everywhere but only continuous at irrational points.

Remark: $f: \mathbb{R} \to \mathbb{R}$ function.

continuous at x_0 : $\lim_{x \to x_0} f(x) = f(x_0) = f(\lim_{x \to x_0} x)$

right-continuous at x_0 : $\lim_{x \to x_0^+} f(x) = f(x_0)$



$\lim_{x \to x_0} f(x) = a$ if:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$|f(x) - a| < \varepsilon$. $\quad \forall x \in (x_0 - \delta) \cup (x_0 + \delta)$

$\lim_{x \to x_0^+} f(x) = a$ if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$|f(x) - a| < \varepsilon \quad \forall x \in (x_0 + \delta)$

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$ $A \to$ set.

uniformly convergence

$$F(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbb{1}_{(-\infty, x]}(r_i)$$ 函数项级数

$\boxed{\sum_{n=1}^{\infty} u_n = \lim_{N\to\infty}\left(\sum_{n=1}^{N} u_n\right)} \leftarrow \quad \left\{\sum_{n=1}^{N} u_n\right\}_{N=1}^{\infty}$

固定 x

$$\sum_{n=1}^{\infty} u_n(x) =: f(x) \quad \text{"逐点"地定义 } f$$

$\forall \varepsilon>0,\ \exists N\in\mathbb{Z}_{>0},\ \forall n\geq N$

$$\left|\sum_{n=1}^{N} u_n(x) - f(x)\right| < \varepsilon \qquad x\in\mathbb{R}$$

则 $\sum_{n=1}^{\infty} u_n(x)$ 一致收敛

(Weierstrass 判别法)

$\sum_{n=1}^{\infty} u_n(x)$，若 $\underline{|u_n(x)| \leq a_n}$ 且 $\sum_{n=1}^{\infty} a_n$ 收敛，则

$\sum_{n=1}^{\infty} u_n(x)$ 一致收敛

若 $\sum_{n=1}^{\infty} u_n(x)$ 一致收敛，则

$$\lim_{x\to x_0}\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty}\left(\lim_{x\to x_0} u_n(x)\right)$$

Proof: $F(x) = \sum_i \boxed{\frac{1}{2^i}\mathbb{1}_{(-\infty,x]}(r_i)}$

$\forall x\in\mathbb{R}$

$$\sum_i \left|\frac{1}{2^i}\mathbb{1}_{(-\infty,x]}(r_i)\right|. \qquad \mathbb{1} = \begin{cases}1\\0.\end{cases}$$

$$\leq \sum_{i=1}^{\infty}\frac{1}{2^i} = \lim_{N\to\infty}\sum_{i=1}^{N}\left(\frac{1}{2}\right)^i \qquad \sum_{i=1}^{N} q^i = \frac{1-q^N}{1-q}\cdot q$$

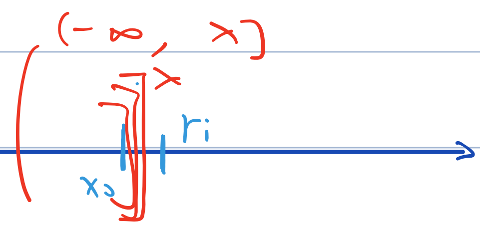
$$= \lim_{N\to\infty}\frac{1}{2}\,\frac{1-\frac{1}{2^N}}{1-\frac{1}{2}} = 1$$

$F(x)$ 一致收敛. 只需判别 $\forall i$ $\frac{1}{2^i}\mathbb{1}_{(-\infty,x]}(r_i)$ 是连续/右连续

$$\lim_{x\to x_0^+}\frac{1}{2^i}\mathbb{1}_{(-\infty,x]}(r_i) \overset{?}{=} \frac{1}{2^i}\mathbb{1}_{(-\infty,x_0]}(r_i)$$

$\| \quad \lim_{x\to x_0^+} f(x) = f(x_0)$

$$\begin{cases}\frac{1}{2^i} & r_i\leq x\\ 0 & r_i > x\end{cases}$$



$x_0 \geq r_i \qquad x_0 < r_i \qquad r_i\in(-\infty, x] \Longrightarrow \underline{\mathbb{1}_{(-\infty,x]}(r_i)=1}$

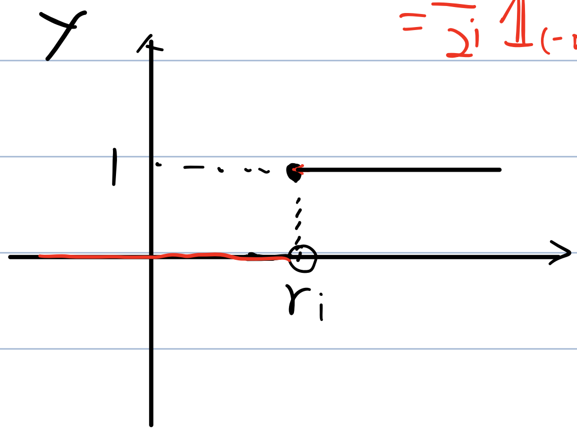
$\mathbb{1}_{[-\infty,x)}(r_i)=1$ $\boxed{\mathbb{1}_{(-\infty,x]}(r_i)=0.}$

$$\lim_{x\to x_0^+}\frac{1}{2^i}\mathbb{1}_{(-\infty,x]}(r_i)=\begin{cases}\frac{1}{2^i} & x_0\geq r_i\\ 0 & x_0<r_i\end{cases}$$

算出来

$\mathbb{1}_A$ 的定义

$$=\frac{1}{2^i}\mathbb{1}_{(-\infty,x_0]}(r_i)$$



$$\mathbb{1}_{(-\infty,x]}(r_i)=\begin{cases}1 & r_i\in(-\infty,x]\\ 0 & r_i\in(x,+\infty)\end{cases}$$

$$=\begin{cases}1 & x\geq r_i\\ 0 & r_i>x\end{cases}$$

对每个 i : $\mathbb{1}_{(-\infty,x]}(r_i)$ 在 $\mathbb{R}\setminus\{r_i\}$ 连续

$F(x)=\sum\frac{1}{2^i}\mathbb{1}_{(-\infty,x]}(r_i)$ 在 $\bigcap_{i=1}^{\infty}(\mathbb{R}\setminus\{r_i\})$ 连续

$$=\mathbb{R}\setminus\bigcup_{i=1}^{\infty}\{r_i\}$$

$$\bigcap_{i=1}^{\infty}\mathbb{R}\setminus\{r_i\}=\bigcap_{i=1}^{\infty}(\mathbb{R}\cap\{r_i\}^c)=\mathbb{R}\cap\left(\bigcap_{i=1}^{\infty}\{r_i\}^c\right)$$

$$=\mathbb{R}\cap\left(\left(\bigcup_{i=1}^{\infty}\{r_i\}\right)^c\right)$$

$\cap A_n^c=(\cup A_n)^c$

$$=\mathbb{R}\cap\mathbb{Q}^c$$

Q 5.

Let f, g be real-value Borel function on $(X,\mathcal{F})$, $A\in\mathcal{F}$.

Show h is a Borel function.

$$h(x)=\begin{cases}f(x) & x\in A\\ g(x) & x\in A^c\end{cases}$$

Remark

$(X,\mathcal{F})$ ⟶ measurable space.

全集 σ- algebra power set 幂集 $\mathcal{P}(X)$

$\mathcal{F}$ 集族 $\to$ $\mathcal{F} \subset \mathcal{P}(X) = \{A : A 为 X 子集\}$

$\mathcal{F} = \{\phi, X\}$

$\mathcal{F}$ is σ - algebra if

① $\phi, X \in \mathcal{F}$.

②. $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$.

③ $A_i \in \mathcal{F}$, $i = 1, 2, \cdots \Longrightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

• $\mathcal{F}$ is σ- algebra, then.

$A_i \in \mathcal{F}, i = 1, \cdots \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Proof: $A_i \in \mathcal{F}. \xRightarrow{②} A_i^c \in \mathcal{F}.$

$\xRightarrow{③} \bigcap_{i=1}^{\infty} A_i^c \in \mathcal{F}$

$\Longrightarrow (\bigcup_{i=1}^{\infty} A_i)^c \in \mathcal{F}$

$\Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

(Real - value).

Borel function : $f : X \longrightarrow \mathbb{R}$ $(X, \mathcal{F})$

f is Borel function if

• $f^{-1}((a,b)) \in \mathcal{F}$ $a, b \in \mathbb{R}$

$= \{x \in X : f(x) \in (a,b)\}$

• $f^{-1}((-\infty, a)) \in \mathcal{F}$ $a \in \mathbb{R}$.

$= \{x \in X : f(x) < a\}$

Proof: for any $(a,b) \subset \mathbb{R}$. it suffices to

$h^{-1}((a,b)) \in \mathcal{F}$

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in A^c. \end{cases}$$

$$h^{-1}((a,b)) = \{x\in X : h(x)\in(a,b)\}$$
$$= \{x\in A : h(x)\in(a,b)\}\cup\{x\in A^c : h(x)\in(a,b)\}$$
$$= \{x\in A : f(x)\in(a,b)\}\cup\{x\in A^c : g(x)\in(a,b)\}$$
$$= (A\cap\{x\in X : f(x)\in(a,b)\})\cup(A^c\cap\{x\in X : g(x)\in(a,b)\})$$
$$= (\underbrace{\underset{\mathcal{F}.}{A}\cap \underset{\mathcal{F}.}{f^{-1}((a,b))}}_{\mathcal{F}.})\cup(\underbrace{\underset{\mathcal{F}}{A^c}\cap \underset{\mathcal{F}.}{g^{-1}((a,b))}}_{\mathcal{F}}) \in\mathcal{F}.$$

Q7

$$\lim_{n\to\infty}\int_1^n \left(1-\frac{t}{n}\right)^n \ln t\,dt \overset{?}{=} \int_1^\infty e^{-t}\ln t\,dt.$$

$$\lim_{n\to\infty}\int_0^1 \left(1-\frac{t}{n}\right)^n \ln t\,dt \overset{?}{=} \int_0^1 e^{-t}\ln t\,dt.$$

Remark Lebesgue dominated convergence

$\{f_n\}$ measurable function.

$f_n : \mathbb{R}^n \to \mathbb{R}.$

$$(X, \mathcal{F}, \mu) \qquad (\mathbb{R}^n, \underset{\mathcal{B}_{\mathbb{R}^n}}{\overset{\parallel}{\mathcal{L}^n}}, m) \qquad \mathbb{R}^n$$

$$(\mathbb{R}, \mathcal{L}^1, m) \qquad \mathbb{R}.$$

$A\in\mathcal{L}^1$, $f^{-1}(A)\in\mathcal{L}^n \to f$ 可测

THM $f, \{f_n\}, g$ measurable function.

$$f_n(x) \to f \quad \text{a.e.}$$

- $|f_n| \le g$
- $\int_{\mathbb{R}^n} g\,dx < \infty$

条件

then: $\lim_{n\to\infty}\int_{\mathbb{R}^n} f_n(x)\,dx = \int_{\mathbb{R}^n}\lim_{n\to\infty} f_n(x)\,dx$ 结论.

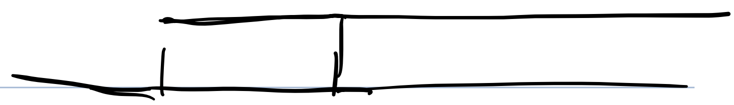
Proof: $\lim_{n\to\infty}\int_1^n \left(1-\frac{t}{n}\right)^n \ln t\,dt$

$$= \lim_{n\to\infty}\int_{\mathbb{R}} \boxed{\left(1-\frac{t}{n}\right)^n \mathbb{1}_{[1,n]}(t)\,\ln t}\,dt \qquad =: f_n$$

$$\int_{\mathbb{R}} \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n \mathbb{1}_{[1,n]}(t)\,\ln t\,dt.$$

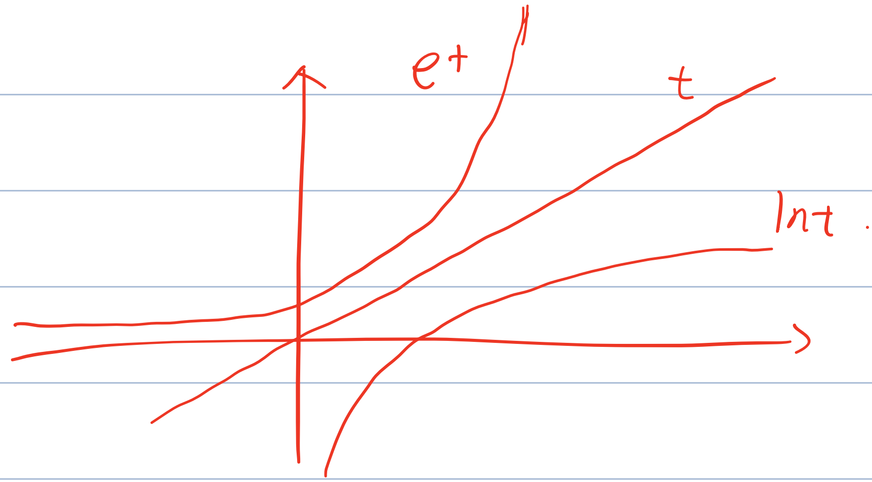
$$= \int_{\mathbb{R}} e^{-t}\,\mathbb{1}_{[1,\infty)}(t)\,\ln t\,dt$$

$$= \int_1^\infty e^{-t}\ln t\,dt$$



$$\left|\left(1-\frac{t}{n}\right)^n \mathbb{1}_{[1,n]}(t)\,\ln t\right| \le e^{-t}\,\mathbb{1}_{[1,\infty)}(t)\,\ln t. = g(t)$$

$\int_1^\infty g(t)\,dt < \infty$ $\qquad$ $\int_1^\infty \frac{1}{t^2}\,dt = -\frac{1}{t}\Big|_1^\infty = 1$



$$\ln t \ll t^\alpha \ll e^t \qquad (t\to\infty)$$

$\frac{\ln t}{e^t}$ “减” 非常快

$\frac{\ln t}{e^t} \ll \frac{1}{t^2}$ 收敛

Q1: 2.3 确界原理

Q2: 10.2 ~ 10.3.

Q8: 3.1 ~ 3.2.

Q5. 1.1

Q6. 1.1 ~ 1.2.

Q7. 3.2.

Q8: 1.3

Q9: 3.3.