Q 1

$\Omega = \mathbb{Q}$, $\mathcal{F}_0 = \left\{ \bigcup_{i=1}^{N} (a_i, b_i] \cap \mathbb{Q} \;:\; a_i < b_i,\ a_i, b_i \in \mathbb{R},\ N \geq 1 \right\}$.

$\boxed{\mathcal{F}_0 \subset 2^{\Omega}}$ $\quad (\Omega, \mathcal{F}) \leftarrow$ measure space.

$\mathcal{F} = \sigma(\mathcal{F}_0)$ $\qquad \mathcal{F}_0 \subset \mathcal{P}(\Omega) = \{ A : A \subset \mathbb{Q} \}$.

$\sigma(\mathcal{F}_0) \subset \boxed{2^{\Omega}} \leftarrow$ 最大的 σ-algebra.

Prove : (a) $\mathcal{F} = 2^{\Omega}$.

(b) If μ is the counting measure, then μ is σ-finite on $\mathcal{F}$, but not on $\mathcal{F}_0$

(c) $A \in \mathcal{F}$ of finite measure cannot be approximated by sets in $\mathcal{F}_0$

(d) If $\lambda = 2\mu$, then $\lambda = \mu$ $\mathcal{F}_0$ but not on $\mathcal{F}$.

Remark: $\Omega \to 2^{\Omega}$, $\mathcal{A} \subset 2^{\Omega}$.

$\sigma(\mathcal{A}) = \bigcap_{\substack{\mathcal{F} \supset \mathcal{A} \\ \mathcal{F} \text{ is a σ-algebra}}} \mathcal{F}$

$\mu(A) = \infty$

$A = \{a_i\}_{i=1}^{\infty}$

$A = \{a_1, \cdots a_N\}$

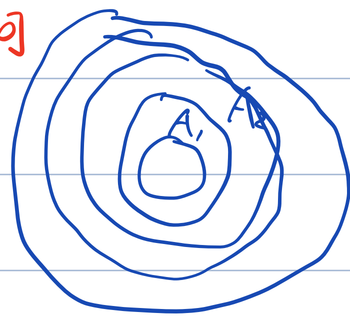
countable measure $\boxed{\mu(A) = \sum_{a \in A} 1}$ $\qquad \mu(A) = N$

σ-finite : $(\underset{\text{全集}}{\Omega}, \underset{\text{σ-代数}}{\mathcal{F}}, \underset{\text{測度}}{\mu})$ $\qquad \mu : \mathcal{F} \to [0, \infty]$

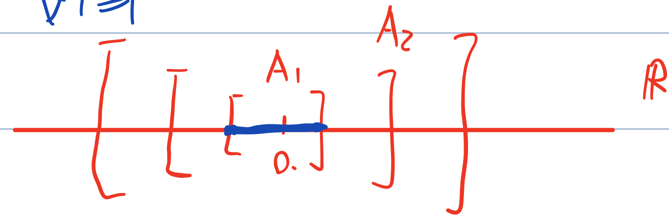
若存在 • $\Omega = \bigcup_{i=1}^{\infty} A_i$ 使得

• $A_i \subset A_{i+1} \quad \forall\, i \geq 1$

• $\mu(A_i) < \infty \quad \forall\, i \geq 1$



例 $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \overset{\mu}{m})$



$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$

$A_n = [-n, n]$ 則 $\quad A_n \subset A_{n+1} \quad \forall\, n \geq 1$

$m(A_n) = m([-n, n]) = 2n < \infty$

$\leadsto$ $\mathbb{R}$ 为 σ-finite.

A can be approximated by set in $\mathcal{F}_0$:

$\exists \{A_n\}_{n=1}^{\infty} \subset \mathcal{F}_0$ s.t. $\limsup A_n = \bigcap_{k\geq 1} \bigcup_{n\geq k} A_n$

$A = \bigcup_{n=1}^{\infty} A_n$

$\mu: \mathcal{F} \longrightarrow [0, \infty]$. $[0, \infty) \cup \{\infty\}$.

$A \longmapsto \mu(A)$ $m(\mathbb{R}) = \infty$

$m([0, \infty)) = \infty$

Proof: $\Omega = \mathbb{Q}$, $\mathcal{F}_0 = \left\{ \bigcup_{i=1}^{N} (a_i, b_i] \cap \mathbb{Q} : \cdots \right\}$.

(a): 要证 $\mathcal{F} = 2^{\Omega}$ $\quad (\mathcal{F} = \sigma(\mathcal{F}_0))$

$\longrightarrow \mathcal{F} \subset 2^{\Omega}$ is obvious. 只需证 $\boxed{2^{\Omega} \subset \mathcal{F}}$ 即可 $\boxed{\forall A \in 2^{\Omega} \Rightarrow A \in \mathcal{F}}$

注意到 $\boxed{\text{只需证 } \forall q \in \mathbb{Q} \text{ 有 } \{q\} \in \mathcal{F}}$ 为什么?

因为 $\forall A \in 2^{\Omega}$, $A \subset \mathbb{Q}$ $A = \{q_i\}_{i=1}^{\infty}$

但 $\{q_i\} \in \mathcal{F}$, $\mathcal{F}$ 为 σ-域, 故 $A = \bigcup_{i=1}^{\infty} \{q_i\} \in \mathcal{F}$ 可数

$\boxed{\forall A \in 2^{\Omega} \text{ 有 } A \in \mathcal{F}}$

$\Downarrow$

$2^{\Omega} \subset \mathcal{F} \xrightarrow{\mathcal{F} \subset 2^{\Omega}} 2^{\Omega} = \mathcal{F}$

下证 $\forall q \in \mathbb{Q}$ 有 $\{q\} \in \mathcal{F}$

事实上有: $\boxed{(q - \frac{1}{n}, q] \cap \mathbb{Q}} \in \mathcal{F}_0 \subset \mathcal{F}$ $(\forall n)$.

$\Downarrow$ $\mathcal{F}$ 为 σ-algebra, 对可列交封闭

$\{q\} = \bigcap_{n=1}^{\infty} (q - \frac{1}{n}, q] \cap \mathbb{Q} \in \mathcal{F}$

(b): $\mu(A) = \sum_{a \in A} 1$

$\Omega = \mathbb{Q} = \bigcup_{i=1}^{\infty} \{r_i\}$ 且 $\{r_i\} \in \mathcal{F}_0$

取 $A_n = \bigcup_{i=1}^{n} \{r_i\} \in \mathcal{F}_0$. 则

- $\mu(A_n) = n < \infty$
- $A_n \subset A_{n+1}$
- $\bigcup A_n = \bigcup_{i=1}^{\infty} \{r_i\} = \mathbb{Q} = \Omega$.

故 μ 在 $\mathcal{F}_0$ 为 σ 有限.

$\forall \bigcup_{i=1}^{N} (a_i, b_i] \cap \mathbb{Q} \in \mathcal{F}_0$, 则

$$\mu\left(\bigcup_{i=1}^{N} (a_i, b_i] \cap \mathbb{Q}\right) = \infty$$ (因为区间有无穷多个元素)

(c) 要证 $A \in \mathcal{F}$, $\mu(A) < \infty$ 不能被 $\mathcal{F}_0$ 中集合逼近.

$\forall A \in \mathcal{F}$, $\mu(A) < \infty$. 则 A 为有限集, 故

$$A = \{a_i\}_{i=1}^{N} = \{a_1, a_2, \dots, a_N\}.$$

但 $\mathcal{F}_0$ 中的元素都形如

$$\bigcup_{i=1}^{N} (a_i, b_i] \cap \mathbb{Q}$$ $\forall B \in \mathcal{F}_0$, $\mu(B) = \infty$

是无限集, 其并不可能有限

反设 $\exists A_n \in \mathcal{F}_0$ 使得 $\bigcup A_n = A$

$\underset{\text{有限}}{A} \neq \underset{\text{无限}}{\bigcup_{n=1}^{\infty} A_n}$, $A_n \in \mathcal{F}_0$

则 $\mu(A) = \mu(\bigcup A_n) \geq \mu(A_1) = \infty$

但 $\mu(A) < \infty$, 矛盾!

规定的另一个测度

(d) 要证: $\lambda = 2\mu$, 则 $\lambda = \mu$ on $\mathcal{F}_0$ but not on $\mathcal{F}$.

$\lambda(A) = 2\mu(A)$ $\forall A \in \mathcal{F}$

$\forall A \in \mathcal{F}_0$, $\mu(A) = \infty$ $\lambda(A) = 2\mu(A) = \infty$

$$\lambda(A) = \mu(A) \quad \forall A \in \mathcal{F}_0$$

即 $\lambda = \mu$ on $\mathcal{F}_0$

取 $A \in \mathcal{F}_0$ 且 $\mu(A) < \infty$，则

$\mu(\{\xi\}) = 1$

$$\boxed{\lambda(A) = 2\mu(A)}$$

即 $\lambda(A) \neq \mu(A)$ if $\mu(A) < \infty$. $\square$.

2. $F: \mathbb{R}^2 \to \mathbb{R}$ right continuous and non decreasing ; ......

$\mathbb{1}(x+y \leq 1)$

3. Let $F: \mathbb{R} \to \mathbb{R}$ be an increasing continuous function and μ be the Lebesgue-Stieltjes measure on $\mathbb{R}$ corresponding F.

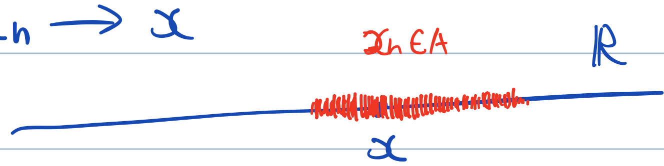
(a) If A is countable subset of $\mathbb{R}$, show that $\mu(A) = 0$

(b) If $\mu(A) > 0$, $\mu(\mathbb{R} \setminus A) = 0$, show A is dense in $\mathbb{R}$.

Remark:

A is dense in $\mathbb{R}$: $\underline{\forall x \in \mathbb{R}}$, $\exists\ x_n \in A\ (n \geq 1)$ s.t.

$x_n \to x$



有理数.

$\mathbb{Q}$ dense in $\mathbb{R}$.

$\sqrt{2} = 1.414\ldots$

1.4

1.41

1.414 ...

A 不在 $\mathbb{R}$ 中稠密：存在 $x_0 \in \mathbb{R}$，$\boxed{x_0 \text{ 不能被 } A \text{ 中元素逼近}}$

$\boxed{\exists\ \varepsilon > 0,\ (x_0 - \varepsilon, x_0 + \varepsilon) \cap A = \phi}$ 不稠密的等价条件

否命题

$\forall \varepsilon > 0$, $(x_0 - \varepsilon, x_0 + \varepsilon) \cap A \neq \phi$ ☆

$\Longrightarrow \forall\ n \geq 1$ $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \cap A \neq \phi$

反证法去证 $\square$ 成立. $\Rightarrow \forall n \geq 1 \quad \exists x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \cap A$

$\Rightarrow \exists \{x_n\} \subset A \quad \boxed{|x_n - x_0| < \frac{1}{n}} \longrightarrow x_n \to x_0$

与 x_0 不能被 A 中元素逼近矛盾!

☆是不成立的 即 $\square$ 成立

Lebesgue - Stieltjes measure.

$F: \mathbb{R} \longrightarrow \mathbb{R}$ function, $\boxed{\text{right-continuous, increasing}}$.

- $(a, b] \longrightarrow \mu((a, b]) = F(b) - F(a).$

$\Downarrow$ $\hookrightarrow$ 半环上的集函数，验证 μ 满足 σ 可加性.

$\Downarrow$ Caratheodory 定理

- $\mu: \mathcal{B}_{\mathbb{R}} \longrightarrow [0, \infty]$. 测度.

$\mu: \mathcal{C} \longrightarrow [0, \infty]$ $\quad (a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$

$\bigcup_{i=1}^{N} (a_i, b_i]$

$\Downarrow$

$\mu: \sigma(\mathcal{C}) \longrightarrow [0, \infty]$

$\parallel$

$\mathcal{B}_{\mathbb{R}}$.

F right continuous, increasing function.

$\Downarrow$ $\quad (a, b]$

μ_F Borel measure

$\Downarrow$

$\mu_F((a, b]) = F(b) - F(a).$

$\Downarrow$

$$\mu_F(A) = \inf \left\{ \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) : A \subset \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}. \quad \forall A \subset \mathbb{R}.$$

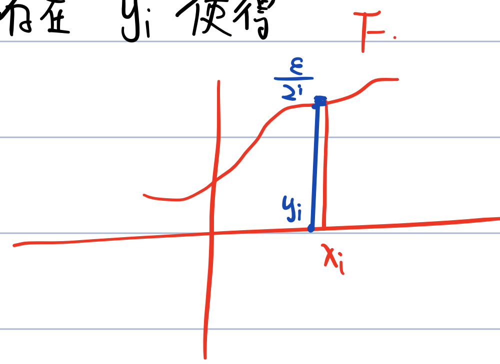
Proof: (a) If A is a countable subset of $\mathbb{R}$, show that: $\mu(A)=0$.

A countable $\Longrightarrow$ $A=\{x_i\}_{i=1}^{\infty}$

$\forall i\geq 1$ $\{x_i\}$单点集 $\forall \varepsilon>0$ 存在 y_i 使得

$$\underline{F(x_i)-F(y_i) < \varepsilon/2^i}$$

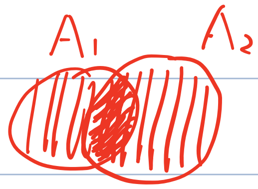
$$\lim_{y\to x_i} F(y) = F(x_i)$$



$$(y_i, x_i] \ni x_i$$

且 $A=\bigcup_{i=1}^{\infty}\{x_i\} \subset \bigcup_{i=1}^{\infty}(y_i, x_i]$

$$\Rightarrow \quad \mu(A) \leq \mu\left(\bigcup_{i=1}^{\infty}(y_i,x_i]\right) \leq \sum_{i=1}^{\infty} \underline{\mu\big((y_i,x_i]\big)} = \sum_{i=1}^{\infty}\big(F(x_i)-F(y_i)\big)$$



$$\mu(A_1\cup A_2) \leq \mu(A_1)+\mu(A_2)$$

$$\leq \sum_{i=1}^{\infty}\frac{\varepsilon}{2^i} = \varepsilon\cdot\sum_{i=1}^{\infty}\frac{1}{2^i} = \varepsilon.$$

即 $\mu(A)<\varepsilon$ 令 $\varepsilon\to 0^+$ 得

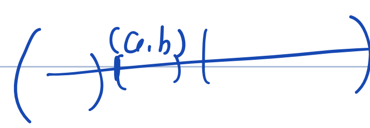
$$0\leq \mu(A)\leq 0$$

即 $\mu(A)=0$

(b) If $\mu(A)>0$, $\underline{\mu(\mathbb{R}\setminus A)=0}$, A is dense in $\mathbb{R}$

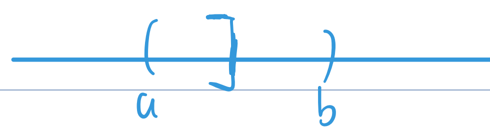
反设 A不在$\mathbb{R}$中稠密, 则 存在 $x_0 \in \mathbb{R}$, $\varepsilon>0$ 使 数分结论(可直接用)

$(a,b) := (x_0-\varepsilon, x_0+\varepsilon)\cap A = \phi$



$\Longrightarrow$ $\underline{(a,b)\subset \mathbb{R}\setminus A}$

$\Longrightarrow$ $\mu\left((a,\frac{a+b}{2}]\right)$



$\leq \mu\big((a,b)\big)$

$\leq \mu(\mathbb{R}\setminus A)=0$

但 $\mu((a,\frac{a+b}{2}]) = F(\frac{a+b}{2}) - F(a) > 0$ Increasing

即 $0 < \mu((a,\frac{a+b}{2}]) \leqslant 0$

矛盾！即 A 在 $\mathbb{R}$ 中稠密. □