Q6.

给定

$\mathcal{A} \subset \mathcal{P}(S)$, $\boxed{E \in \sigma(\mathcal{A})}$. Show that $\exists\ \mathcal{A}_0 \subset \mathcal{A}$, $\mathcal{A}_0$ countable s.t. : $E \in \sigma(\mathcal{A}_0)$

Remark

$\mathcal{A} \subset \mathcal{P}(S)$ 集族（一族集合）

$\sigma(\mathcal{A})$: 包含 $\mathcal{A}$ 的最小的 σ-代数

$$\sigma(\mathcal{A}) \overset{\triangle}{=} \bigcap_{\substack{\mathcal{A} \subset \mathcal{F} \\ \mathcal{F}\text{为}\sigma\text{代数}}} \mathcal{F}. \qquad \mathcal{A} \subset \boxed{\mathcal{F}} \ \bigcap$$

$\boxed{\mathcal{P}(S)}$

☆ · $\mathcal{G}$: σ-代数, $\mathcal{A} \subset \mathcal{G} \Rightarrow \sigma(\mathcal{A}) \subset \mathcal{G}$. ☆

$$\sigma(\mathcal{A}) = \bigcap_{\substack{\mathcal{A} \subset \mathcal{F} \\ \mathcal{F}\text{为}\sigma\text{代数}}} \mathcal{F} \subset \mathcal{G}$$

$$\bigcap_{n=1}^{\infty} [1,n] \subset [1,m]$$

$\hookrightarrow \{[1,n] : n \geq 1\} \ni [1,m]$

$A = B \iff A \subset B$ 且 $B \subset A$

$\iff \begin{cases} \forall x \in A \text{ 有 } x \in B \ ☆ \\ \forall x \in B \text{ 有 } x \in A. \ ☆ \end{cases}$

Proof: "单调类定理" Monotone class theorem.

$$\mathcal{G} = \{E \in \sigma(\mathcal{A}) ; \exists\ \mathcal{A}\text{ 的可数子类 } \mathcal{A}_0,\ \text{使得 } E \in \sigma(\mathcal{A}_0)\}.$$

如果 $\boxed{\mathcal{G} = \sigma(\mathcal{A})}$, 则

$\forall\ E \in \sigma(\mathcal{A})$ 有 $E \in \mathcal{G}$ → $\mathcal{G}$的定义

↳ 即 存在 $\mathcal{A}$ 的可数子类 $\mathcal{A}_0$ 使得 $E \in \sigma(\mathcal{A}_0)$

$\sigma(\mathcal{A}) \supset \mathcal{G}$ 显然. $\qquad A = B \iff A \subset B$ 且 $B \subset A$

只需证明 $\sigma(\mathcal{A}) \subset \mathcal{G}$

$E \in \mathcal{A} \Rightarrow E \in \mathcal{G}$

- 验证：$\mathcal{A} \subset \mathcal{G}$

$\mathcal{G}$的定义

$E_{\exists} \Leftrightarrow E \in \mathcal{G}$

$\forall E \in \mathcal{A}$，验证存在 $\mathcal{A}$ 的可数子类 $\mathcal{A}_0$ 使得 $E \in \sigma(\mathcal{A}_0)$

取 $\mathcal{A}_0 = \{E\}$，可数，且 $E \in \sigma(\mathcal{A}_0)$

故 $E \in \mathcal{G}$

- 验证 $\mathcal{G}$ 为 σ-代数

$\phi \in \sigma(\mathcal{A}_0)$

① 验证 $\phi, \Omega \in \mathcal{G}$

找 $\mathcal{A}_0 \to$ 可数

任取 $E \in \sigma(\mathcal{A})$，取 $\mathcal{A}_0 = \{E\}$，

$\phi, \Omega \in \sigma(\mathcal{A}_0) \leftarrow$ σ代数定义

$\mathcal{G} = \{E \in \sigma(\mathcal{A}):$ 存在 $\mathcal{A}$ 可数子类 $\mathcal{A}_0$，$E \in \sigma(\mathcal{A}_0)\}$

ϕ 找到一个 $\mathcal{A}_0$，$\phi \in \sigma(\mathcal{A}_0)$

$\Rightarrow \phi, \Omega \in \mathcal{G}$

② 验证 $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$. √

$\forall A \in \mathcal{G}$，$\mathcal{G} = \{E \in \sigma(\mathcal{A}): \exists \mathcal{A}$ 可数子类 $\mathcal{A}_0$, $E \in \sigma(\mathcal{A}_0)\}$

则存在 $\mathcal{A}$ 可数子类 $\mathcal{A}_0$ 使得

$A \in \sigma(\mathcal{A}_0)$ $\rightarrow$ σ-代数性质

进一步： $A^c \in \sigma(\mathcal{A}_0)$

$A^c \to$ 找到一个 $\mathcal{A}_0$ s.t. $A^c \in \sigma(\mathcal{A}_0)$

$\Downarrow$

$A^c \in \mathcal{G}$

③ 验证可列交封闭

即: $\boxed{E_i \in \mathcal{G} \quad i=1,\cdots \Longrightarrow \bigcap_{i=1}^{\infty} E_i \in \mathcal{G}}$

任给 $E_i \in \mathcal{G}$ $i=1,2,\cdots$ ，则

$E_i \in \mathcal{G} \Longleftrightarrow$ 存在 $\mathcal{A}$ 的可数子类 $\mathcal{A}_i$ 使 $\boxed{E_i \in \sigma(\mathcal{A}_i)}$

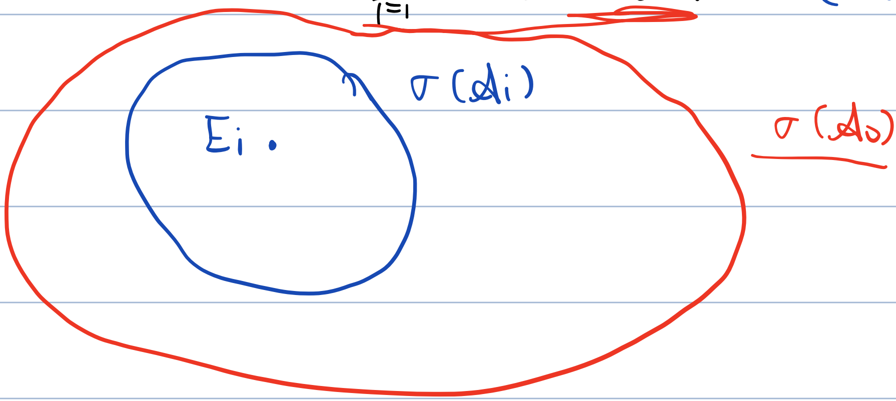
取 $\mathcal{A}_0 = \bigcup_{i=1}^{\infty} \mathcal{A}_i \rightsquigarrow$ 可数

$$\sigma(\mathcal{A}_i) \subset \sigma(\mathcal{A}_0) = \sigma\left(\bigcup_{i=1}^{\infty} \mathcal{A}_i\right)$$

则 $E_i \in \sigma(\mathcal{A}_i) \subset \sigma(\mathcal{A}_0)$ $\forall$ i 都成立.

又 $\sigma(\mathcal{A}_0)$ 为 σ-代数，故

$\bigcap_{i=1}^{\infty} E_i \in \sigma(\mathcal{A}_0)$ （对可列交封闭）



$\bigcap_{i=1}^{\infty} E_i \longrightarrow$ 找到了可数子列 $\mathcal{A}_0$ ，使 $\bigcap_{i=1}^{\infty} E_i \in \sigma(\mathcal{A}_0)$

$\Longleftrightarrow \bigcap_{i=1}^{\infty} E_i \in \mathcal{G}$

综上: $\mathcal{G}$ 是 σ-algebra. 且 $\boxed{\mathcal{A} \subset \mathcal{G}}$

$\Longrightarrow \sigma(\mathcal{A}) \subset \mathcal{G}$ $\square$

Q8

Let $\{\mu_n\}$ be an increasing sequence of measures on $(X, \mathcal{F})$

i.e. $\mu_{n+1}(A) \geq \mu_n(A)$ $\forall A \in \mathcal{F}$.

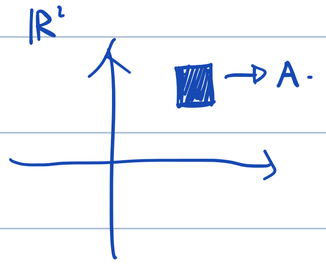
Suppose. $\boxed{\lim_{n\to\infty} \mu_n(A) = \mu(A) \quad \forall A \in \mathcal{F}}$, Show μ is a measure.

Remark.

$(X, \mathcal{F})$

measure $\quad \mu: \mathcal{F} \longrightarrow [0,+\infty]$

$\qquad\qquad A \longmapsto \mu(A)$



$\mu(A) \longrightarrow$ A的面积 $\longrightarrow$ $\mathbb{R}^2$ 中的 Lebesguse.

$\mu: \mathcal{F} \longrightarrow [0,\infty]$ is $\boxed{\text{measure}}$ if

① $\mu(\phi) = 0$.

② 若 $\{A_i\}_{i=1}^{\infty} \subset \mathcal{F}$ 且 $A_i \cap A_j = \phi$ $(i \neq j)$ 则

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Proof:

① $\boxed{\mu(\phi)} = \lim_{n\to\infty} \mu_n(\phi) = \lim_{n\to\infty} 0 \boxed{= 0}$.

measure

$\in \mathcal{F}$.

② A_i 不交集列 要证 $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

即证: $\lim_{n\to\infty} \mu_n\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lim_{n\to\infty} \mu_n(A_i)$

即证 $\lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i) = \sum_{i=1}^{\infty} \lim_{n\to\infty} \mu_n(A_i)$

• $\boxed{\sum_{i=1}^{\infty} \mu(A_i) < \infty}$

$A = B \iff A \geq B$ 且 $A \leq B$

极限

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i)$$

不等 $\quad \mu_n(A) \leq \mu_m(A) \leq \mu(A)$

$$\leq \lim_{n\to\infty} \boxed{\sum_{i=1}^{\infty} \mu(A_i)}$$

与n无关

$$= \sum_{i=1}^{\infty} \mu(A_i).$$

即: $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$

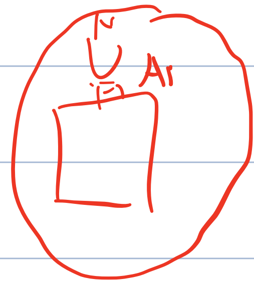
下证 $\boxed{\mu\left(\bigcup_{i=1}^{\infty} A_i\right)} \geq \sum_{i=1}^{\infty} \mu(A_i)$

$\sum_{i=1}^{\infty} \mu(A_i) < \infty \Rightarrow \forall \varepsilon > 0, \exists N \geq 1$ 使

$\sum_{i=1}^{N} \mu(A_i) > \sum_{i=1}^{\infty} \mu(A_i) - \varepsilon$.

有限

$$\begin{aligned}
\sum_{i=1}^{N} \mu(A_i) &= \sum_{i=1}^{N} \lim_{n\to\infty} \mu_n(A_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{N} \mu_n(A_i) \\
&= \lim_{n\to\infty} \mu_n\left(\bigcup_{i=1}^{N} A_i\right) \\
&\leq \boxed{\lim_{n\to\infty} \mu_n}\left(\bigcup_{i=1}^{\infty} A_i\right) \\
&= \mu\left(\bigcup_{i=1}^{\infty} A_i\right)
\end{aligned}$$



$\bigcup_{i=1}^{\infty} A_i$

即有: $\sum_{i=1}^{\infty} \mu(A_i) - \varepsilon < \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$

令 $\varepsilon \to 0^+$ 有

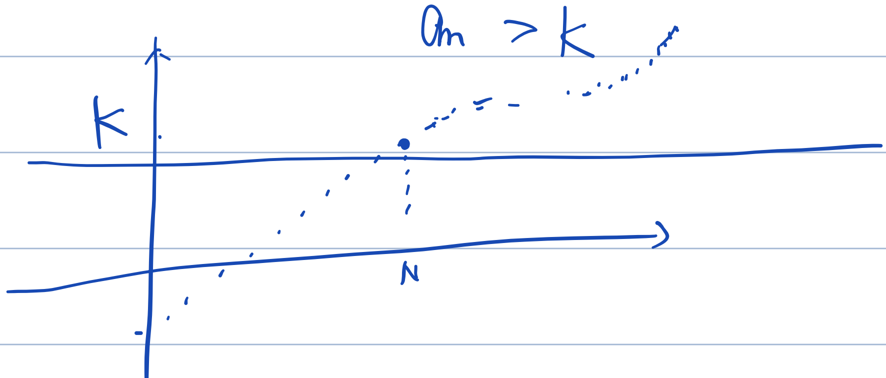
$$\sum_{i=1}^{\infty} \mu(A_i) \leq \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$$

- $\boxed{\sum_{i=1}^{\infty} \mu(A_i) = +\infty}$

Remark: $\lim_{n\to\infty} a_n = +\infty$

$\forall K > 0$, $\exists N \in \mathbb{Z}_{>0}$ 使得对任意 $n > N$ 都有

$a_n > K$



$\forall K > 0$, $\exists N \in \mathbb{Z}_{>0}$, $\forall m > N$

$$\begin{aligned}
K &< \sum_{i=1}^{m} \mu(A_i) \\
&= \sum_{i=1}^{m} \lim_{n\to\infty} \mu_n(A_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{m} \mu_n(A_i) \\
&= \lim_{n\to\infty} \mu_n\left(\bigcup_{i=1}^{m} A_i\right)
\end{aligned}$$

$$\leq \lim_{n\to\infty} \mu_n\left(\bigcup_{i=1}^{\infty} A_i\right)$$

$$= \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$$

令 $k \to +\infty$ 有

$$+\infty \leq \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq +\infty.$$

即 $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty = \sum_{i=1}^{\infty} \mu(A_i)$ $\square$.