$T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$

- $T^*T \in \mathcal{L}(\mathcal{H}_1)$
- T^*T self-adjoint
- T^*T nonnegative definite.

$\forall x \in \mathcal{H}_1 \qquad \langle T^*Tx, x\rangle$

$= \langle Tx, Tx\rangle = \|Tx\| \geq 0.$

By theorem 3.4.3. we can defined the square root of T^*T. $(T^*T)^{\frac{1}{2}}$

- $(T^*T)^{\frac{1}{2}} \in \mathcal{L}(\mathcal{H}_1)$
- $(T^*T)^{\frac{1}{2}}$ self adjoint.
- $(T^*T)^{\frac{1}{2}}$ nonnegative definite.
- $\left((T^*T)^{\frac{1}{2}}\right)^2 = T^*T$.

$$\left((T^*T)^{\frac{1}{2}}\right)^{\frac{1}{2}} =: (T^*T)^{\frac{1}{4}}.$$

Def : $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{H}_1$, $\mathcal{H}_2$ separable Hilbert space.
T is trace class operator if for some CONS $\{e_j\}_{j=1}^{\infty}$ of $\mathcal{H}_1$.

$$\|T\|_{TR} = \sum_{j=1}^{\infty} \langle (T^*T)^{\frac{1}{2}} e_j, e_j\rangle \quad (*) .$$

$\|T\|_{TR}$. is said to be the trace norm of T.

$$\langle (T^*T)^{\frac{1}{2}} e_j, e_j\rangle = \langle \underbrace{(T^*T)^{\frac{1}{4}}}_{((T^*T)^{\frac{1}{4}})^*} \cdot (T^*T)^{\frac{1}{4}} e_j, e_j\rangle.$$

$$= \langle (T^*T)^{\frac{1}{4}} e_j, (T^*T)^{\frac{1}{4}} e_j\rangle.$$

$$\|T\|_{TR} = \sum_{j=1}^{\infty} \langle (T^*T)^{\frac{1}{2}} e_j, e_j\rangle$$

$$= \sum_{j=1}^{\infty} \langle (T^*T)^{\frac{1}{4}} e_j, (T^*T)^{\frac{1}{4}} e_j\rangle$$

$$= \|(T^*T)^{\frac{1}{4}}\|_{HS}^2.$$

$\Rightarrow$ (*) 式不依赖于 CONS 的选取.

T is trace class operator.

Since. $\|(T^*T)^{\frac{1}{4}}\|_{HS}^2 = \|T\|_{TR} < \infty$.

$$(T^*T)^{\frac{1}{4}} \in \mathcal{L}_{HS}(\mathcal{H}_1) \subset C(\mathcal{H}_1)$$

$$\Rightarrow ((T^*T)^{\frac{1}{4}})^4 = T^*T \in C(\mathcal{H}_1).$$

Let $T^*T = \sum \lambda_j^2 e_j \otimes e_j$.

$$\lambda_1 \geq \lambda_2 \geq \cdots$$

$$\Rightarrow (T^*T)^{1/2} = \sum \sqrt{\lambda_j^2}\, e_j \otimes e_j$$

$\{\varepsilon_j\}$ 为 CONS. $\lambda_j \rightsquigarrow$ T的奇异值.

$\varepsilon_j = e_j$

$$\|T\|_{TR} = \sum_{j=1}^{\infty} \langle (T^*T)^{1/2} \varepsilon_j, \varepsilon_j \rangle$$

$$= \sum_{j=1}^{\infty} \langle \sum_{i=1}^{\infty} \lambda_i (e_i \otimes e_i) \varepsilon_j, \varepsilon_j \rangle$$

$$= \sum_{j=1}^{\infty} \langle \sum_{i=1}^{\infty} \lambda_i \langle \varepsilon_j, e_i \rangle e_i, \varepsilon_j \rangle$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \lambda_i \langle \varepsilon_j, e_i \rangle^2.$$

$\lambda_i \langle \varepsilon_j, e_i \rangle^2 \geq 0 \quad \forall i, j.$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \lambda_i \langle e_i, \varepsilon_j \rangle^2.$$

$\sum_{j=1}^{\infty} \langle e_i, \varepsilon_j \rangle^2 = \|e_i\|^2 = 1$

$$= \sum_{j=1}^{\infty} \lambda_i$$

$\|T\|_{TR} = \sum_{i=1}^{\infty} \lambda_i$. $\lambda_i \rightsquigarrow$ T的奇异值.

TR ~ HS $\qquad \lambda_1 \geq \lambda_2 \geq \cdots$

$$\|T\|_{HS}^2 = \sum_{i=1}^{\infty} \lambda_i^2 \leq \sum_{i=1}^{\infty} \lambda_1 \cdot \lambda_i = \lambda_1 \sum_{i=1}^{\infty} \lambda_i = \lambda_1 \|T\|_{TR} < \infty.$$

If T is trace class, then T is HS operator.

(THM 4.5.2).

Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. T is self-adjoint and trace class. Let $\{\lambda_j\}_{j=1}^{\infty}$ be the eigenvalue sequence of T. then.

$$T = \sum_{j=1}^{\infty} \lambda_j e_j \otimes e_j.$$

$$\|T\|_{TR} = \sum \langle (T^*T)^{1/2} e_j, e_j \rangle.$$

$$(T^*T)^{1/2} = (T^2)^{1/2} = \sum_{j=1}^{\infty} \sqrt{\lambda_j^2}\, e_j \otimes e_j$$

$$= \sum_{j=1}^{\infty} |\lambda_j|\, e_j \otimes e_j.$$

CONS $\to \{e_j\}$.

$$\|T\|_{TR} = \sum_{j=1}^{\infty} |\lambda_j| \quad (<\infty).$$

We can define $\qquad \left|\sum_{j=1}^{\infty} \lambda_j\right| \leq \sum_{j=1}^{\infty} |\lambda_j| < \infty.$

$$\operatorname{trace}(T) = \sum_{j=1}^{\infty} \lambda_j.$$

Proposition. $\quad \operatorname{trace}(T) = \sum_{j=1}^{\infty} \langle T e_j, e_j \rangle.$

for all $\{e_j\}_{j=1}^{\infty}$ CONS.

Proof. $\quad T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$

$\forall \{\varepsilon_j\}$ CONS.

$$\sum_{j=1}^{\infty} \langle T\varepsilon_j, \varepsilon_j \rangle = \sum_{j=1}^{\infty} \left\langle \left(\sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i\right) \varepsilon_j, \varepsilon_j \right\rangle.$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \lambda_i \langle e_i, \varepsilon_j \rangle^2.$$

$$\|T\|_{TR} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\lambda_i| \langle e_i, \varepsilon_j \rangle^2 < \infty.$$

$$= \sum_{i=1}^{\infty} \lambda_i \sum_{j=1}^{\infty} \langle e_i, \varepsilon_j \rangle^2 \quad = \|e_i\|^2 = 1.$$

$$= \sum_{i=1}^{\infty} \lambda_i.$$

$\operatorname{trace}(T) = \sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{\infty} \langle T e_i, e_i \rangle.$ for all CONS $\{e_i\}$.

$\|T\|_{TR} = \sum_{i=1}^{\infty} |\lambda_i| = \sum_{i=1}^{\infty} \langle (T^*T)^{1/2} e_i, e_i \rangle.$ for all CONS $\{e_i\}$.

If T is nonnegative, then.

$$\lambda_i = |\lambda_i|$$

In this case. $\|T\|_{TR} = \operatorname{trace}(T)$.

In finite dimenisonal case

$$\operatorname{trace}(T) = \sum_{j=1}^{N} \lambda_j$$

$$\operatorname{trace}(aT_1 + bT_2) = a\operatorname{trace}(T_1) + b\operatorname{trace}(T_2).$$

$$\operatorname{trace}(T_1T_2) = \operatorname{trace}(T_2T_1).$$

Theorem. Suppose $T, \tilde{T}$ are HS operators form $\mathcal{H}_1$ to $\mathcal{H}_2$ with nonascending singular value $\lambda_j, \tilde{\lambda}_j$ $(j \geq 1)$, then.

$$\sum_{j=1}^{\infty} (\lambda_j - \tilde{\lambda}_j)^2 \leq \operatorname{trace}\left((T-\tilde{T})^*(T-\tilde{T})\right).$$

Proof.

Let.

$$T = \sum_{j=1}^{\infty} \lambda_j f_{1j} \otimes f_{2j}$$

$$\tilde{T} = \sum_{j=1}^{\infty} \tilde{\lambda}_j \tilde{f}_{1j} \otimes \tilde{f}_{2j}$$

$$\boxed{\begin{aligned} \lambda_j f_{2j} &= T f_{1j} \\ \tilde{\lambda}_j \tilde{f}_{2j} &= \tilde{T}\tilde{f}_{1j}. \end{aligned}}$$

Then

$$\operatorname{trace}\left((T-\tilde{T})^*(T-\tilde{T})\right)$$

$$= \sum_{j=1}^{\infty} \langle (T-\tilde{T})^*(T-\tilde{T}) f_{1j}, f_{1j} \rangle$$

$$= \sum_{j=1}^{\infty} \langle (T-\tilde{T}) f_{1j}, (T-\tilde{T}) f_{1j} \rangle \quad (= \langle T-\tilde{T}, T-\tilde{T} \rangle_{HS}).$$

$$= \sum_{j=1}^{\infty} \left(\langle T f_{1j}, T f_{1j} \rangle - 2\langle T f_{1j}, \tilde{T} f_{1j} \rangle + \langle \tilde{T} f_{1j}, \tilde{T} f_{1j} \rangle \right).$$

$$= \sum_{j=1}^{\infty} \lambda_j^2 + \sum_{j=1}^{\infty} \tilde{\lambda}_j^2 - 2 \sum_{j=1}^{\infty} \langle T f_{1j}, \tilde{T} f_{1j} \rangle.$$

$$T f_{1j} = \lambda_j f_{2j}.$$

$$\tilde{T} f_{1j} = \left(\sum_{i=1}^{\infty} \tilde{\lambda}_i \tilde{f}_{1i} \otimes \tilde{f}_{2i} \right) f_{1j} = \sum_{i=1}^{\infty} \tilde{\lambda}_i \langle \tilde{f}_{1i}, f_{1j} \rangle \tilde{f}_{2i}.$$

$$\sum \langle T f_{1j}, \tilde{T} f_{1j} \rangle. = \sum_{i=1}^{\infty} \tilde{\lambda}_i \lambda_j \langle \tilde{f}_{1i}, f_{1j} \rangle \langle \tilde{f}_{2i}, f_{2j} \rangle.$$

$$\mathrm{trace}((T-\tilde{T})^*(T-\tilde{T}))$$
$$=\sum_{j=1}^{\infty}\lambda_j^2+\sum_{j=1}^{\infty}\tilde{\lambda}_j^2-2\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_j\tilde{\lambda}_i\langle\tilde{f}_{1i},f_{1j}\rangle\langle\tilde{f}_{2i},f_{2j}\rangle.$$

$\forall j\geq 1$
$$\left|\sum_{i=1}^{\infty}\langle\tilde{f}_{1i},f_{1j}\rangle\langle\tilde{f}_{2i},f_{2j}\rangle\right|$$
$$\leq\left(\sum_{i=1}^{\infty}\langle\tilde{f}_{1i},f_{1j}\rangle^2\right)^{1/2}\left(\sum_{i=1}^{\infty}\langle\tilde{f}_{2i},f_{2j}\rangle^2\right)^{1/2}.$$
$$=(\|f_{1j}\|^2)^{1/2}(\|f_{2j}\|^2)^{1/2}\quad=1.$$
$\forall i\geq 1.$
$$\left|\sum_{j=1}^{\infty}\langle\tilde{f}_{1i},f_{1j}\rangle\langle\tilde{f}_{2i},f_{2j}\rangle\right|\leq 1$$
(*)

要证：
$$\sum_{j=1}^{\infty}(\lambda_j-\tilde{\lambda}_j)^2\leq\mathrm{trace}(T-\tilde{T})^*(T-\tilde{T})$$
$$=-2\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_j\tilde{\lambda}_i a_{ji}+\sum_{j=1}^{\infty}\lambda_j^2+\sum_{j=1}^{\infty}\tilde{\lambda}_j^2.$$

where $a_{ji}=\langle\tilde{f}_{1i},f_{1j}\rangle\langle\tilde{f}_{2i},f_{2j}\rangle$

只需证.
$$\sum_{j=1}^{\infty}(\lambda_j-\tilde{\lambda}_j)^2=\sum_{j=1}^{\infty}\lambda_j^2+\sum_{j=1}^{\infty}\tilde{\lambda}_j^2-2\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_j\tilde{\lambda}_i$$
$$\leq\sum_{j=1}^{\infty}\lambda_j^2+\sum_{j=1}^{\infty}\tilde{\lambda}_j^2-2\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_j\tilde{\lambda}_i a_{ji}.$$

只需证.
$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_j\tilde{\lambda}_i\geq\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_j\tilde{\lambda}_i a_{ji}.$$

$N\to\infty.$

只需证
$$\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_j\tilde{\lambda}_i-\sum_{j=1}^{N}\sum_{i=1}^{N}\lambda_j\tilde{\lambda}_i a_{ji}\geq 0.$$

$$T_N=\sum_{j=1}^{N}\lambda_j f_{1j}\otimes f_{2j}$$

$$\tilde{T}_N=\sum_{j=1}^{N}\tilde{\lambda}_j\tilde{f}_{1j}\otimes\tilde{f}_{2j}.$$

由(*).有
$$\left|\sum_{i=1}^{N}a_{ji}\right|\leq 1$$
$$\left|\sum_{j=1}^{N}a_{ji}\right|\leq 1.$$

令 $\lambda=(\lambda_1,\cdots\lambda_N)^T\qquad A=(a_{ji})_{N\times N}.$

$$\tilde{\lambda} = (\tilde{\lambda}_1, \cdots \tilde{\lambda}_N)^T$$

只需证：$\underline{\lambda^T \lambda \geqslant \lambda^T A \lambda}$. 对 j 任意的 N 成立.

考虑 $f(N) = \lambda^T\lambda - \lambda^T A \lambda$. $(\geqslant 0)$

Note that. $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$

$\tilde{\lambda}_1 \geqslant \tilde{\lambda}_2 \geqslant \cdots$

Let. $\alpha_j = (1, \cdots \underset{\uparrow\, j}{1}, 0, \cdots 0)^T$.

Then $\lambda = \sum_{j=1}^{N} k_j \alpha_j \qquad k_j \geqslant 0$

$\tilde{\lambda} = \sum_{j=1}^{N} \tilde{k}_j \alpha_j \qquad \tilde{k}_j \geqslant 0$.

$$f(N) = \lambda^T \tilde{\lambda} - \lambda^T A \tilde{\lambda}$$

$$= \sum_{j=1}^{N}\sum_{i=1}^{N} k_j \tilde{k}_i \alpha_j^T \alpha_i - \sum_{j=1}^{N}\sum_{i=1}^{N} k_j \tilde{k}_i \alpha_j^T A \alpha_i$$

$$= \underline{\sum_{i=1}^{N}\sum_{i=1}^{N} k_j \tilde{k}_i}\ \underline{\alpha_j^T (I - A)\alpha_i}$$

即若 $\forall\, k, l \geqslant 1$

$$\alpha_k^T (I - A) \alpha_l \geqslant 0.$$

则 $f(N) \geqslant 0$. 不妨设 $k \geqslant l$. 则.

$$\alpha_k^T \alpha_l = (\underbrace{1 \cdots 1}_{k}, 0 \cdots 0) \left(\begin{matrix} \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \right) l.$$

$$\underline{= l.}$$

$$\alpha_k^T A \alpha_l = (\underbrace{1, \cdots 1}_{k}, 0 \cdots 0) \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \\ a_{1N} & \cdots & a_{NN} \end{pmatrix} \left(\begin{matrix} \vdots \\ 0 \\ \vdots \\ 0 \end{matrix} \right) l.$$

$$= \sum_{j=1}^{k}\sum_{i=1}^{l} a_{ji}.$$

$$= \sum_{i=1}^{l}\left(\sum_{j=1}^{k} a_{ji}\right)$$

从而. $\alpha_k^T \alpha_l - \alpha_k^T A \alpha_l$

$$= \sum_{i=1}^{\ell}\left(1 - \sum_{j=1}^{k} a_{ji}\right) \geq 0.$$

$\Rightarrow$ $f(N) \geq 0$. 令 $N \to \infty$.

Let $(E, \mathcal{B}, \mu)$ be a measure space for some finite measure μ.
i.e. $\mu(E) < \infty$.

Suppose K is measure on $E \times E$ such that.

$$\int_{E\times E} K^2(s,t)\, d\mu(s)\, d\mu(t) < \infty.$$

and define the integral operator $\mathcal{K}$ by

$$\mathcal{K}f(\cdot) = \int_E K(s,\cdot) f(s)\, d\mu(s) \qquad \forall f \in \mathbb{L}^2(E,\mathcal{B},\mu).$$

The function K is referred as the kernel of $\mathcal{K}$.

By Cauchy-schwarz inequality

$$\|\mathcal{K}f\|^2 = \int_E |\mathcal{K}f(t)|^2\, d\mu(t)$$

$$= \int_E \left|\int_E K(s,t) f(s)\, d\mu(s)\right|^2 d\mu(t).$$

$$\leq \int_E \left(\int_E K^2(s,t)\, d\mu(s) \int_E f^2(s)\, d\mu(s)\right) d\mu(t)$$

$$= \|f\|^2 \left(\int_{E\times E} K^2(s,t)\, d\mu(s)\, d\mu(t)\right).$$

$\Rightarrow$

$$\|\mathcal{K}f\| \leq \underbrace{\left(\int_{E\times E} K^2(s,t)\, d\mu(s)\, d\mu(t)\right)^{1/2}}_{} \|f\|.$$

$$\|\mathcal{K}f\| \leq M\|f\|$$

$\Rightarrow$ $\mathcal{K} \in \mathcal{L}(\mathbb{L}^2(E,\mathcal{B},\mu))$ and.

$$\|\mathcal{K}\| \leq \left(\int_{E\times E} K^2(s,t)\, d\mu(s)\, d\mu(t)\right)^{1/2}.$$

$\{$ E : compact metric space.

$\mathcal{B} = \mathcal{B}(E)$ Borel σ- field of E.
$\text{supp}(\mu) = E$
K: continuous on $E \times E$.

$(kf)(\cdot) = \int_E K(s, \cdot) f(s) \, d\mu(s)$

Lemma 4.6.1 $\forall f \in L^2$. $(kf)(\cdot)$ is uniformly continuous

Proof. E compact $\Rightarrow$ $E \times E$ compact.

As K is continuous on $E \times E$, then. K is uniformly continuous on $E \times E$.

$\forall \varepsilon > 0$, $\exists \delta > 0$, such that.

$$|t - t'| < \delta \Rightarrow \underline{|K(s,t) - K(s,t')| < \varepsilon}.$$

$$|(kf)(t) - (kf)(t')|$$

$$= \left| \int_E K(s,t) f(s) \, d\mu(s) - \int_E K(s,t') f(s) \, d\mu(s) \right|$$

$$\leq \int_E |K(s,t) - K(s,t')| \, |f(s)| \, d\mu(s).$$

$$\leq \left(\int_E |K(s,t) - K(s,t')|^2 d\mu(s) \right)^{1/2} \left(\int_E |f(s)|^2 d\mu(s) \right)^{1/2}.$$

$$\leq \left(\varepsilon^2 \mu(E) \right)^{1/2} \cdot \|f\| = \underline{\mu(E)^{1/2}} \, \underline{\|f\|} \cdot \varepsilon.$$

$\Rightarrow$. $(kf)(\cdot)$ is uniformly continuous.

Theorem. $\underline{k \text{ is compact.}}$

Proof By Stone - Weierstrass theorem, $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{Z}_{>0}$ and $\{g_i\}_{i=1}^{n_\varepsilon}$ $\{h_i\}_{i=1}^{n_\varepsilon} \subset C(E)$, such that

$$\sup_{s,t \in E} |K(s,t) - K_{n_\varepsilon}(s,t)| < \underline{\varepsilon}.$$

Where $K_{n_\varepsilon}(s,t) = \sum_{i=1}^{n_\varepsilon} g_i(s) h_i(t)$

Define k_{n_ε} be the integral operation with kernel K_{n_ε}

$$(k_{n_\varepsilon} f)(\cdot) = \int_E K_{n_\varepsilon}(s,\cdot) f(s)\, d\mu(s)$$

$$= \sum_{i=1}^{n_\varepsilon} \int_E h_i(\cdot)\, g_i(s) f(s)\, d\mu(s).$$

$$= \sum_{i=1}^{n_\varepsilon} \int_E g_i(s) f(s)\, d\mu(s)\, h_i(\cdot).$$

$\Rightarrow$. $\quad \mathrm{Im}(k_{n_\varepsilon}) \subset \mathrm{span}\{h_1, \cdots h_{n_\varepsilon}\}.$

$\Rightarrow$ rank $(k_{n_\varepsilon}) \leq n_\varepsilon.$

$\Rightarrow$ k_{n_ε} is compact.

$$\|(k_{n_\varepsilon} - k) f\|^2 = \int_E \left| \int_E (K_{n_\varepsilon}(s,t) - K(s,t)) f(s)\, d\mu(s) \right|^2 d\mu(t).$$

$$\leq \int_E \left(\int_E |K_{n_\varepsilon}(s,t) - K(s,t)|^2 d\mu(s) \int_E f(s)^2\, d\mu(s) \right) d\mu(t)$$

$$\leq \|f\|^2\, \varepsilon^2 \cdot \mu(E)^2.$$

$f \in \{ f \in L^2 : \|f\| \leq 1 \} =: \mathcal{F}$

$$\|k_{n_\varepsilon} - k\| = \sup_{f \in \mathcal{F}} \|(k_{n_\varepsilon} - k) f\|$$

$$\leq \varepsilon \cdot \mu(E)$$

$\Rightarrow$ k is compact.

Recall Example 3.3.4.

K symmetry $\Rightarrow$ k self-adjoint.

$(K(s,t) = K(t,s))$.

By theorem 4.2.4.

$$\mathscr{k} = \sum_{j=1}^{\infty} \lambda_j e_j \otimes e_j.$$

And.

$$\mathscr{k} e_j = \lambda_j e_j \Rightarrow e_j(t) = \frac{1}{\lambda_j} \int_E K(s,t) e_j(s) \, d\mu(s).$$

$$\Downarrow$$

uniformly continuous

By Lemma 4.6.1

$e_j(t)$ uniformly continuous.

Example. $K(s,t) = \min(s,t)$.

$E = [0,1]$.

$$\begin{aligned}(\mathscr{k} f)(t) &= \int_0^1 K(s,t) f(s) \, ds. \\ &= \int_0^1 \min(s,t) f(s) \, ds. \\ &= \int_0^t s f(s) \, ds + \int_t^1 t f(s) \, ds. \\ &= \int_0^t s f(s) \, ds + t \int_t^1 f(s) \, ds.\end{aligned}$$

Eigenvalue : $(\mathscr{k} e)(t) = \lambda e(t)$.

$$\Downarrow$$

$$\int_0^t s e(s) \, ds + t \int_t^1 e(s) \, ds = \lambda e(t). \quad ①.$$

$$t e(t) + \int_t^1 e(s) \, ds - t e(t) = \lambda e'(t)$$

$$\Rightarrow \int_t^1 e(s) \, ds = \lambda e'(t). \quad ②.$$

$$\Rightarrow -e(t) = \lambda e''(t)$$

$$\Rightarrow e''(t) + \frac{1}{\lambda} e(t) = 0.$$

① Let $t=0$: $e(0) = 0$.

② Let $t=1$ $e'(1) = 0$

$$e(t) = a \sin\left(\frac{t}{\sqrt{\lambda}}\right) + b \cos\left(\frac{t}{\sqrt{\lambda}}\right).$$

$e(0) = b = 0.$

$e'(t) = a\frac{1}{\sqrt{\lambda}}\cos\left(\frac{t}{\sqrt{\lambda}}\right) - \frac{b}{\sqrt{\lambda}}\sin\left(\frac{t}{\sqrt{\lambda}}\right)$

$e'(1) = a\frac{1}{\sqrt{\lambda}}\cos\left(\frac{1}{\sqrt{\lambda}}\right) = 0.$

$\frac{1}{\sqrt{\lambda}} = \frac{(2j-1)\pi}{2} \qquad j \geq 1.$

$$\lambda_j = \frac{4}{((2j-1)\pi)^2}. \rightsquigarrow e_j(t) = \sqrt{2}\sin\left(\frac{(2j-1)\pi}{2}t\right)$$

Recall $K(x,y)$ on $E\times E$ is nonnegative definite if.

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j K(t_i,t_j) \geq 0.$$

$\forall \{a_i\} \subset \mathbb{R}$, $\{t_i\} \subset E$, $m \geq 1$

Theorem. An integral operator is nonnegative definite iff its kernel is nonnegative definite.

Proof. Let $K(s,t)$ be the kernel of $\mathcal{K}$.

"$\Leftarrow$" Suppose $K(s,t)$ is nonnegative definite.

$K(s,t)$. uniformly continuous on $E\times E$. $\forall n \geq 1$ $\exists \delta_n > 0$ such that

$$d((s_1,t_1),(s_2,t_2)) < \delta_n \Rightarrow |K(s_1,t_1) - K(s_2,t_2)| < \frac{1}{n}.$$

Since E is compact mertice space.

$$E = \bigcup_{i=1}^{N(n)} E_{n_i} \qquad \operatorname{diam}(E_{n_i}) < \delta_n.$$

$\forall 1 \leq i \leq N(n)$ choose $v_i \in E_{n_i}$ define

$$K_n(s,t) = \sum_{i=1}^{N(n)}\sum_{j=1}^{N(n)} K(v_i,v_j)\chi_{E_{n_i}}(s)\chi_{E_{n_j}}(t)$$

then.

$$\max_{(s,t)\in E\times E} |K_n(s,t) - K(s,t)| < \frac{1}{n}. \checkmark$$

$(\langle \mathcal{K}f, f\rangle \geq 0)$

$$|\langle \mathcal{K}f, f\rangle - \langle \mathcal{K}_n f, f\rangle|$$

$$= \left|\int_E \left(\int_E (K(s,t) - K_n(s,t))f(s)\,d\mu(s)\right) f(t)\,d\mu(t)\right|$$

$$\leq \int_E \int_E \underline{|K(s,t) - K_n(s,t)|}\, f(s)\, d\mu(s)\, f(t)\, d\mu(t)$$

$$\leq \frac{1}{n} \int_E |f(s)|\, d\mu(s) \int_E |f(t)|\, d\mu(t)$$

$$= \frac{1}{n} \left(\int_E |f(s)|\, d\mu(s) \right)^2$$

$$\leq \frac{1}{n} \int_E |f(s)|^2 d\mu(s) \cdot \int_E 1^2 d\mu(s) = \mu(E)\, \|f\|^2 \cdot \frac{1}{n}.$$

$$\Rightarrow \quad \underbrace{\langle k_n f, f \rangle}_{\geq 0.} \to \langle k f, f \rangle$$

$$\langle k_n f, f \rangle = \int_E \int_E \sum_{i=1}^{N(n)} \sum_{j=1}^{N(n)} K(v_i, v_j)\, f(s)\, \chi_{E_{n_i}}(s)\, \chi_{E_{n_j}}(t)\, d\mu(s) \; f(t)\, d\mu(t)$$

$$= \sum_{i=1}^{N(n)} \sum_{j=1}^{N(n)} K(v_i, v_j) \underbrace{\int_{E_{n_i}} f(s)\, d\mu(s)}_{a_i} \underbrace{\int_{E_{n_j}} f(t)\, d\mu(t)}_{a_j}.$$

$$\geq 0.$$

Since $K(x,y)$ is nonnegative ditinite.

Letting $n \to \infty$.

$$\langle K f, f \rangle \geq 0. \qquad \forall f \in L^2.$$

"$\Rightarrow$" suppose that.

$$\underline{\sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j K(v_i, v_j) < 0.} \;\; (\geq 0) \qquad \times$$

for some $\{a_i\} \subset \mathbb{R}$, $\{v_i\} \subset E$.

As K is uniform continuous. $\exists \{E_i\} \subset \mathcal{B}$. with.

$$v_i \in E_i, \qquad \mu(E_i) > 0. \qquad \forall i.$$

$$\underline{\max_{x_i, y_i \in E_i} \sum_i \sum_j a_i a_j K(x_i, y_i) < 0.}$$

$$\underline{\sum_i \sum_j a_i a_j \frac{1}{\mu(E_i)\mu(E_j)} \int_{E_i} \int_{E_j} K(u,v)\, d\mu(u)\, d\mu(v).}$$

$$= \sum_i \sum_j a_i a_j K(\xi_i, \eta_i). \quad < 0.$$

$\underline{\xi_i, \eta_i \in E_i}$.

$$f = \sum_i \frac{a_i}{\mu(E_i)} \chi_{E_i}.$$

$$\langle kf, f\rangle = \int_E \int_E k(x,y) f(x)\, d\mu(y) f(y) d\mu(x)$$

$$= \int_E \int_E k(x,y) \sum_i \sum_j \frac{a_i}{\mu(E_i)} \frac{a_j}{\mu(E_j)} \underline{\chi_{E_i}}(x) \chi_{E_j}(y)\, d\mu(x) d\mu(y)$$

$$= \sum_i \sum_j \frac{a_i a_j}{\mu(E_i)\mu(E_j)} \int_{E_i}\int_{E_j} k(x,y)\, d\mu(x)\, d\mu(y)$$

$< 0.$

Theorem 4.6.5

self-adjoint nonnegative definite ($\mathcal{k}$).

Let the continuous kernel k be symmetric and nonnegative definite and $\mathcal{k}$ be the corresponding integral kernel.

If $\{(\lambda_j, e_j)\}_{j=1}^{\infty}$ are eigenvalue and eigenfunction pair of $\mathcal{k}$.

$$[\, \mathcal{k} = \sum_{j=1}^{\infty} \lambda_j\, e_j \otimes e_j \,]$$

then.

$$\boxed{k(s,t) = \sum_{j=1}^{\infty} \lambda_j e_j(s)\, e_j(t).}$$

Where the sum converging absolutely and uniformly on E.

Lemma.

1. $\sum_{j=1}^{\infty} \lambda_j e_j^2(t) \le k(t,t)$ $\quad (\forall t \in E)$

2. $\sum_{j=1}^{\infty} |\lambda_j e_j(s) e_j(t)| \le (k(t,t)\, k(s,s))^{1/2}$ $\quad (\forall\ t,s \in E)$ $\quad < \infty$

3. $\lim_{n\to\infty} \sup_{s,t} \sum_{j=n+1}^{\infty} |\lambda_j e_j(s) e_j(t)| = 0.$

4. the function $\sum_{j=1}^{\infty} \lambda_j e_j(s) e_j(t)$ is $\boxed{\text{well defined}}$ and uniformly continuous with the sum converging $\boxed{\text{absolutely}}$ and $\boxed{\text{uniformly}}$.

Proof.

$\mathcal{k}$.

① Let $k_n(s,t) = k(s,t) - \sum_{j=1}^{n} \lambda_j e_j(s) e_j(t).$

$[\; K_n(t,t) = K(t,t) - \sum_{j=1}^{n} \lambda_j e_j^2(t) \geq 0 \; (\forall n) \;]$

and $\mathcal{K}_n$ be the integral operator with kernel K_n: $\forall f \in L^2$.

$$\langle \mathcal{K}_n f, f\rangle = \langle \mathcal{K} f, f\rangle - \sum_{j=1}^{n} \lambda_j \langle f, e_j\rangle^2.$$

$$= \sum_{j=1}^{\infty} \lambda_j \langle f, e_j\rangle^2 - \sum_{j=1}^{n} \lambda_j \langle f, e_j\rangle^2.$$

$\mathcal{K} = \sum_{j=1}^{\infty} \lambda_j e_j \otimes e_j$

$$= \sum_{j=n+1}^{\infty} \underbrace{\lambda_j}_{\geq 0.} \langle f, e_j\rangle^2. \quad \geq 0.$$

It follows that $\mathcal{K}_n$ is nonnegative-definite.

$\Rightarrow$ K_n is nonnegative-definite.

$\Rightarrow$. $\underline{K_n(t,t) \geq 0.}$ $\forall t \in E$.

Let $n \to \infty$. $\lim_{n\to\infty} K_n(t,t) = \lim_{n\to\infty} (K(t,t) - \sum_{j=1}^{n} \lambda_j e_j^2(t))$

$$= K(t,t) - \sum_{j=1}^{\infty} \lambda_j e_j^2(t) \geq 0.$$

② $\forall n \geq 1$. $K_n(t,t) = \underline{K(t,t) - \sum_{j=1}^{n} \lambda_j e_j^2(t)}$

$$\sum_{j=1}^{n} |\lambda_j e_j(s) e_j(t)| = \sum_{j=1}^{n} |\sqrt{\lambda_j} e_j(s) \sqrt{\lambda_j} e_j(t)|$$

$$\leq \left(\sum_{j=1}^{n} \lambda_j e_j^2(s)\right)^{1/2} \left(\sum_{j=1}^{n} \lambda_j e_j^2(t)\right)^{1/2}.$$

$$\leq (K(s,s) K(t,t))^{1/2}.$$

Let $n \to \infty$.

$$\sum_{j=1}^{\infty} |\lambda_j e_j(s) e_j(t)| \leq (K(s,s) K(t,t))^{1/2}.$$

③. $\forall s, t \in E$.

$$\sum_{j=1}^{\infty} \lambda_j e_j^2(s) \leq K(s,s) < \infty. \quad \sum_{j=1}^{\infty} \lambda_j e_j^2(t) \leq K(t,t) < \infty.$$

$\Rightarrow$. $\sum_{j=n+1}^{\infty} \lambda_j e_j^2(s) \to 0$. $\quad \sum_{j=n+1}^{\infty} \lambda_j e_j^2(t) \to 0$. (as $n \to \infty$)

$$\sum_{j=n+1}^{\infty} |\lambda_j e_j(s) e_j(t)| \leq \left(\sum_{j=n+1}^{\infty} \lambda_j e_j^2(s)\right)^{1/2} \left(\sum_{j=n+1}^{\infty} \lambda_j e_j^2(t)\right)^{1/2}.$$

$\to 0.$ (as $n\to\infty$).

i.e. $\sum_{j=n+1}^{\infty} |\lambda_j e_j(s) e_j(t)| \to 0$ pointwise.

- $|\lambda_j e_j(s) e_j(t)|$ continuous in E.
- $|\lambda_j e_j(s) e_j(t)| \geq 0$. pointwise.
- E compact.
- 0 continuous in E.

By Dini theorem.

$$\sum_{j=n+1}^{\infty} |\lambda_j e_j(s) e_j(t)| \to 0 \quad \text{uniformly}$$

i.e. $\underline{\lim_{n\to\infty} \sup_{s,t} \sum_{j=n+1}^{\infty} |\lambda_j e_j(s) e_j(t)|} = 0.$

④. $\forall \varepsilon>0$, $\exists N\in\mathbb{Z}_{>0}$ s.t.

$$\sup_{s,t} \sum_{j=N+1}^{\infty} |\lambda_j e_j(s) e_j(t)| < \varepsilon$$

$$\Rightarrow \quad \sum_{j=N+1}^{\infty} |\lambda_j e_j(s) e_j(t)| < \varepsilon \qquad \forall\ s,t\in E.$$

By the uniformly continuous of $\{e_j\}_{j=1}^{N}$, $\exists \delta>0$. s.t.

$$d((s,t),(s',t'))<\delta \Rightarrow \left|\sum_{j=1}^{N}\lambda_j e_j(s,t) - \sum_{j=1}^{N}\lambda_j e_j(s',t')\right|<\varepsilon.$$

$$\Rightarrow \quad \underline{\left|\sum_{j=1}^{\infty}\lambda_j e_j(s,t) - \sum_{j=1}^{\infty}\lambda_j e_j(s',t')\right|}$$

$$\leq \left|\underline{\sum_{j=1}^{N}\lambda_j e_j(s,t) - \sum_{j=1}^{N} e_j(s',t')}\right| + \sum_{j=N+1}^{\infty}\left(\underline{|\lambda_j e_j(s,t)|} + \underline{|\lambda_j e_j(s',t')|}\right).$$

$$\leq \varepsilon + \varepsilon + \varepsilon$$

$$= 3\varepsilon \qquad d((s,t),(s',t'))<\delta.$$

<u>Proof of ~~THM 4.6.5~~</u>. kernel operator.

$$K(s,t) \rightsquigarrow \mathcal{K}.$$

$$\tilde{K}(s,t) = \sum_{j=1}^{\infty}\lambda_j e_j(s) e_j(t) \rightsquigarrow \tilde{\mathcal{K}}$$

$$\forall f\in L^2 \quad \mathcal{K}f = \left(\sum \lambda_j e_j\otimes e_j\right) f$$

$$= \sum \lambda_j \langle f, e_j \rangle e_j$$

$$= \int \sum \lambda_j e_j(s) e_j(\cdot) f(s)\, d\mu(s).$$

$$= \tilde{k} f.$$

$\Rightarrow$. $\quad k = \tilde{k}$. $\quad \boxed{k = \tilde{k}}$.

If $K \neq \tilde{K}$. suppose $(s_0, t_0) \in E \times E$ with.

$$K(s_0, t_0) \neq \tilde{K}(s_0, t_0).$$

By continuity of $K, \tilde{K}$, $\exists\, E_0 \in \mathcal{B}$ with,

$\mu(E_0) > 0$. $\quad K(s,t) - \tilde{K}(s,t).\ (>0)\ (<0)$. $\quad \forall\, s \in E_0$.

$s_0 \in E_0$.

Let $f = \chi_{E_0}$, then.

$$\int_E K(s,t) f(s)\, d\mu(s) - \int_E \tilde{K}(s,t) f(s)\, d\mu(s).\ (>0)\ (<0).$$

i.e, $\qquad \underset{\shortparallel}{k f} \neq \underset{\shortparallel}{\tilde{k} f}$

But $k = \tilde{k}$.

$$\Rightarrow. \quad K(s,t) = \tilde{K}(s,t) = \sum_{j=1}^{\infty} \lambda_j e_j(s) e_j(t).$$

Theorem. Under condition of THM 4.6.5.

$$\operatorname{trace}(k) = \int_E K(s,s)\, d\mu(s).$$

$$\boxed{\|k\|_{HS}^2 = \int_E \int_E K^2(s,t)\, d\mu(s)\, d\mu(t).}$$

Proof.

$$\operatorname{trace}(k) = \sum_{j=1}^{\infty} \lambda_j = \sum_{j=1}^{\infty} \lambda_j \int_E e_j^2(t)\, d\mu(t)$$

$$= \int_E \sum_{j=1}^{\infty} \lambda_j e_j^2(t)\, d\mu(t)$$

$$= \int_E K(t,t)\, d\mu(t).$$

Let $\quad K_n(s,t) = \sum_{j=1}^{n} \lambda_j e_j(s) e_j(t). \rightsquigarrow k_n$

Then. $\quad k_n = \sum_{j=1}^{n} \lambda_j e_j \otimes e_j.$

$$k_n^2 = \sum_{j=1}^{n} \lambda_j^2 e_j \otimes e_j.$$

$\Rightarrow$ $\quad \mathrm{trace}(k_n^2) = \sum_{j=1}^{n} \lambda_j^2 \longrightarrow \sum_{j=1}^{\infty} \lambda_j^2 = \|k\|_{HS}^2$

$\forall f \in L^2$. $\quad k_n^2 f(\cdot) = k_n(k_n f)(\cdot)$

$$= \int_E K_n(s,\cdot)(k_n f)(s)\,d\mu(s).$$

$$= \int_E K_n(s,\cdot)\left(\int_E K_n(t,s) f(t)\,d\mu(t)\right) d\mu(s)$$

$$= \int_E \underbrace{\left(\int_E K_n(s,\cdot) K_n(t,s)\,d\mu(s)\right)}_{T(t,\cdot)} f(t)\,d\mu(t)$$

$$= \int_E T(t,\cdot) f(t)\,d\mu(t)$$

$$\mathrm{trace}(k_n^2) = \int_E T(t,t)\,d\mu(t).$$

$$= \int_E \int_E K_n(s,t) K_n(t,s)\,d\mu(s)\,d\mu(t)$$

$$= \int_E \int_E K_n^2(s,t)\,d\mu(s)\,d\mu(t)$$

- $K_n^2(s,t) \longrightarrow K^2(s,t)$. $\quad \forall\, s,t$
- $K^2(s,t) \leq K(s,s) K(t,t)$.
- $\int_E \int_E K(s,s) K(t,t)\,d\mu(s)\,d\mu(t) = \mathrm{trace}(k)^2 < \infty$.

By dominated convergence theorem,

$$\int_E \int_E K_n^2(s,t)\,d\mu(s)\,d\mu(t) \longrightarrow \int_E \int_E K^2(s,t)\,d\mu(s)\,d\mu(t)$$

$$\| \quad \|k\|_{HS}^2$$

Kernel $K(s,t)$. has rank r if the corresponding integral operator has rank r.

Theorem. Let K be a symmetric and nonnegative-definite kernel with eigen decomposition

$$K(s,t) = \sum_{j=1}^{\infty} \lambda_j e_j(s) e_j(t).$$

Then $\forall r \in \mathbb{Z}_{>0}$. for with. $\lambda_r > 0$.

$$\min_{\mathrm{rank}(W)=r} \iint_{E\times E} (K(s,t) - W(s,t))^2 \, d\mu(s)\, d\mu(t).$$

$$= \sum_{j=r+1}^{\infty} \lambda_j^2 .$$

where the minimum is achieved by.

$$W(s,t) = \sum_{j=1}^{r} \lambda_j e_j(s) e_j(t).$$

Proof. Let. $\mathcal{k} \rightsquigarrow K$.

$\mathcal{W} \rightsquigarrow W$.

By THM 4.4.7

$$\min \|\mathcal{k} - \mathcal{W}\|_{HS}^2 = \sum_{j=r+1}^{\infty} \lambda_j^2 .$$

$$\Rightarrow \quad \mathcal{W} = \sum_{j=1}^{r} \lambda_j e_j \otimes e_j$$

Kernel $W(s,t) = \sum_{j=1}^{r} \lambda_j e_j(s) e_j(t).$

$$\min \|\mathcal{k} - \mathcal{W}\|_{HS}^2 = \min \iint_{E\times E} (K(s,t) - W(s,t))^2 \, d\mu(s)\, d\mu(t).$$