Regularization parameter selection.

$\mathcal{Y} = \mathbb{R}^n$, $\quad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, $\quad \mathcal{H}_0 = \operatorname{span}\{\phi_1, \cdots \phi_q\}$.

$$\|Y\|_{\mathcal{Y}}^2 = \frac{1}{n}\sum_{j=1}^{n} Y_j^2$$

Ordinary cross validation.

$m_\eta^{[k]}$ minimizer of.

$$\frac{1}{n}\sum_{\substack{1\le i\le n\\ i\ne k}} (Y_i - T_i g)^2 + \eta \langle g, P_1 g\rangle_{\mathcal{H}} . \qquad g \in \mathcal{H}$$

$(Y_k - T_k g)^2 \frac{1}{n}$. $\quad m_\eta$

define.

$$CV(\eta) = \frac{1}{n}\sum_{k=1}^{n} (Y_k - T_k m_\eta^{[k]})^2 .$$

$Y_k - T_k m_\eta$

pick η.

Def. $\forall y \in \mathcal{Y}$, let $m_\eta[k, y]$ be the minimizer of.

$$\frac{1}{n}(y - T_k g)^2 + \frac{1}{n}\sum_{\substack{1\le i\le n\\ i\ne k}} (Y_i - T_i g)^2 + \eta\langle g, P_1 g\rangle_{\mathcal{H}} \quad (g \in \mathcal{H})$$

Recall THM.6.4.1

$$f(g; T, P_1, Y, \eta) = \|Y - Tg\|_{\mathcal{Y}}^2 + \eta\langle g, P_1 g\rangle_{\mathcal{H}}, \qquad (g \in \mathcal{H}). \qquad (*)$$

$Tg = (T_1 g, \cdots T_n g)^T \qquad T_i g = \langle \tau_i, g\rangle \qquad \xi_i = P_1 \tau_i$

$\Phi = (T_i \phi_j)_{n\times q} \qquad \Psi = (T_i \xi_j)_{n\times n} = (\langle \xi_i, \xi_j\rangle)_{n\times n}.$

$\mathcal{U} = (\Phi \ \Psi). \qquad \mathcal{V} = \begin{pmatrix} O_{q\times q} & O_{q\times n} \\ O_{n\times q} & \Psi_{n\times n} \end{pmatrix}$

If $\mathcal{U}^T\mathcal{U} + \eta\mathcal{V}$ is invertible.

$$m_\eta = \sum_{j=1}^{q} \hat{c}_j \phi_j + \sum_{j=1}^{n} \hat{b}_j \xi_j$$

$$= (\phi_1, \cdots \phi_q, \xi_1, \cdots \xi_n)\begin{pmatrix}\hat{c}\\ \hat{b}\end{pmatrix}. \qquad \hat{c} = \begin{pmatrix}\hat{c}_1\\ \vdots \\ \hat{c}_q\end{pmatrix}. \quad \hat{b} = \begin{pmatrix}\hat{b}_1\\ \vdots\\ \hat{b}_n\end{pmatrix}.$$

$$\begin{pmatrix}\hat{c}\\ \hat{b}\end{pmatrix} = (\mathcal{U}^T\mathcal{U} + \eta\mathcal{V})^{-1}\mathcal{U}^T Y.$$

$$\frac{1}{n}\sum_{1\le i\le n}(Y_i - T_i g)^2 + \eta\langle g, P_1 g\rangle_{\mathcal{H}} \quad \leftarrow \quad m_\eta.$$

$$\frac{1}{n}(\underset{\sim}{y} - T_k g)^2 + \frac{1}{n}\sum_{\substack{1\le i\le n\\ i\ne k}}(Y_i - T_i g)^2 + \eta\langle g, P_1 g\rangle_{\mathcal{H}} \quad \leftarrow \quad m_\eta[k, y].$$

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \rightsquigarrow \quad m_\eta.$$

$$\parallel$$

$$Y^{[k]} := \begin{pmatrix} Y_1 \\ \vdots \\ Y_{k-1} \\ y \\ Y_{k+1} \\ \vdots \\ Y_n \end{pmatrix} \rightsquigarrow \quad \underline{m_\eta[k, y]}.$$

Y_k.

$Y^{[k]}$.

Take $y = Y_k$

$$m_\eta[k, Y_k] = m_\eta \qquad \forall k.$$

Lemma. $m_\eta[k, \underbrace{T_k m_\eta^{[k]}}_{Y_k'}] = \underline{m_\eta^{[k]}}$.

__Proof__.

It suffices to check that.

$$\underline{\frac{1}{n}(\underline{T_k m_\eta^{[k]} - T_k m_\eta^{[k]}})^2 + \sum_{\substack{1\le i\le n\\ i\ne k}}(Y_i - T_i m_\eta^{[k]})^2 + \eta\langle m_\eta^{[k]}, P_1 m_\eta^{[k]}\rangle_{\mathcal{H}}}$$

$$= \sum_{\substack{1\le i\le n\\ i\ne k}}(Y_i - T_i m_\eta^{[k]})^2 + \eta\langle m_\eta^{[k]}, P_1 m_\eta^{[k]}\rangle_{\mathcal{H}}.$$

$$\le \underline{\sum_{\substack{1\le i\le n\\ i\ne k}}(Y_i - T_i g)^2 + \eta\langle g, P_1 g\rangle_{\mathcal{H}}}$$

$$\textcircled{\le} \ \frac{1}{n}\left(T_k m_n^{[k]} - T_k g\right)^2 + \sum_{\substack{1\le j\le n \\ j\ne k}} (Y_j - T_j g) + \eta \langle g, g \rangle_{\mathcal{H}}.$$

$(\forall g)$.

i.e. $m_n[k, T_k m_n^{[k]}] = m_n^{[k]}$. $\square$

define.

$$a_{kk}(n) = \frac{T_k m_n - T_k m_n^{[k]}}{Y_k - T_k m_n^{[k]}} = \frac{T_k m_n - Y_k + Y_k - T_k m_n^{[k]}}{Y_k - T_k m_n^{[k]}}$$

$$= \frac{T_k m_n - Y_k}{Y_k - T_k m_n^{[k]}} + 1.$$

$$\Rightarrow \quad Y_k - T_k m_n^{[k]} = \frac{Y_k - T_k m_n}{1 - a_{kk}(n)}.$$

$$\Rightarrow \quad CV(n) = \frac{1}{n}\sum_{k=1}^{n}\left(Y_k - T_k m_n^{[k]}\right)^2.$$

$$\boxed{= \frac{1}{n}\sum_{k=1}^{n} \frac{(Y_k - T_k m_n)^2}{(1 - a_{kk}(n))^2}}$$

m_n v.
a_{kk}?

Let $Y_k' = T_k m_n^{[k]}$. then.

$$m_n[k, Y_k'] = m_n^{[k]}.$$

$$m_n[k, Y_k] = m_n.$$

$$a_{kk}(n) = \frac{T_k m_n - T_k m_n^{[k]}}{Y_k - T_k m_n^{[k]}} = \frac{T_k m_n[k, Y_k] - T_k m_n[k, Y_k']}{Y_k - Y_k'}$$

$T_k m_n[k, y]$ ← linear

$$T_k m_n[k, y] = (T_k\phi_1, \cdots T_k\phi_q, T_k\xi_1, \cdots T_k\xi_n)(\mathcal{U}^T\mathcal{U} + \eta\mho)^{-1}\mathcal{U}. \begin{pmatrix} Y_1 \\ \vdots \\ Y_{k-1} \\ y \\ Y_{k+1} \\ \vdots \\ Y_n \end{pmatrix}$$

$$\mathcal{U} = (\Phi, \Psi). \quad \Phi = (T_i\phi_j)_{n\times q_0} \quad \Psi = (T_i\xi_j)_{n\times n}.$$

$$T_k m_\eta[k, y] = e_k^T \mathcal{U}(\mathcal{U}^T\mathcal{U} + \eta\mathcal{V})^{-1}\mathcal{U}\Big(\sum_{\substack{1\le i\le n\\ i\ne k}} Y_i e_i + \underline{y e_k}\Big).$$

$$\frac{\partial}{\partial y} T_k m_\eta[k, y] = e_k^T \underline{\mathcal{U}(\mathcal{U}^T\mathcal{U} + \eta\mathcal{V})^{-1}\mathcal{U}} e_k$$

$$a_{kk}(\eta). = e_k^T \underline{A(\eta)} e_k.$$

$$A(\eta) = \mathcal{U}(\mathcal{U}^T\mathcal{U} + \eta\mathcal{V})^{-1}\mathcal{U} \rightsquigarrow a_{kk}(\eta). \rightsquigarrow CV(\eta).$$

$$Y_k - T_k m_\eta^{[k]}.$$

Generalized cross validation.

$$GCV(\eta) = \frac{\|Y - Tm_\eta\|_Y^2}{(1 - \frac{1}{n}\operatorname{trace} A(\eta))^2} = \frac{1}{n}\frac{\sum_{k=1}^n (Y_k - T_k m_\eta)^2}{(1 - \frac{1}{n}\operatorname{trace} A(\eta))^2}$$

$$= \frac{1}{n}\frac{\sum_{k=1}^n (Y_k - T_k m_\eta^{[k]})^2 (1 - a_{kk}(\eta))^2}{(1 - \frac{1}{n}\operatorname{trace} A(\eta))^2}.$$

Let. $W_{kk}(\eta) = \dfrac{(1 - a_{kk}(\eta))^2}{(1 - \frac{1}{n}\operatorname{trace} A(\eta))^2}$. Then.

$$GCV(\eta) = \frac{1}{n}\sum_{k=1}^n (Y_k - T_k m_\eta^{[k]})^2 \underline{W_{kk}(\eta)}.$$

$$CV(\eta) = \frac{1}{n}\sum_{k=1}^n (Y_k - T_k m_\eta^{[k]})^2.$$

Assume: $\varepsilon = (\varepsilon_1, \cdots \varepsilon_n)^T$ $\underline{\mathbb{E}\varepsilon_j = 0}$. $\underline{\operatorname{Var}\varepsilon_j = \sigma^2}$ for all $j: 1\le j\le n$.

Let.

$$\mu = \mathbb{E}Y. = (T_1 m, \cdots, T_n m)^T$$

$$\operatorname{Risk}(\eta) = \mathbb{E}\|A(\eta)Y - \mu\|_Y^2 = \frac{1}{n}.\mathbb{E}(A(\eta)Y - \mu)^T(A(\eta)Y - \mu)$$

As.

$$\mathbb{E}(\underline{A(\eta)}Y-\mu) = A(\eta)\,\mathbb{E}Y-\mu = (A(\eta)-I)\,\mu.$$

$$\mathbb{V}\text{ar}\,(A(\eta)Y) = \mathbb{E}\,(A(\eta)Y - \underbrace{\mathbb{E}A(\eta)Y}_{A(\eta)\mu})^T(A(\eta)Y-\mathbb{E}A(\eta)Y)$$

$$= \mathbb{E}\,(A(\eta)(Y-\mu))^T(A(\eta)(Y-\mu))$$

$$= \underline{\mathbb{E}\ \varepsilon^T A(\eta)^T A(\eta)\,\varepsilon}.$$

$$= \mathbb{E}\left(\sum_{i,j}\varepsilon_i\varepsilon_j\, e_i^T A(\eta)^T A(\eta)\, e_j\right).$$

$$\begin{cases} i\neq j. \\ \qquad \mathbb{E}\,\varepsilon_i\varepsilon_j = \mathbb{E}\varepsilon_i\,\mathbb{E}\varepsilon_j = 0. \\ i=j. \\ \qquad \mathbb{E}\,\varepsilon_i\varepsilon_j = \mathbb{E}\,\varepsilon_i^2 = \mathbb{E}(\varepsilon_i-0)^2 = \mathbb{V}\text{ar}\,\varepsilon_i = \sigma^2. \end{cases}$$

$$\Rightarrow.\quad \mathbb{V}\text{ar}\,(A(\eta)Y) = \sum_{1\le i\le n}\sigma^2\, e_i^T \underline{A(\eta)^T A(\eta)}\, e_i$$

$$= \sigma^2\ \text{trace}\,(A(\eta)^T A(\eta)).$$

$$\text{Risk}(\eta) = \mathbb{E}\,\|A(\eta)Y-\mu\|_Y^2.$$

$$= \mathbb{E}\,\|A(\eta)Y - A(\eta)\mu + A(\eta)\mu - \mu\|_Y^2.$$

$$= \mathbb{E}\,\|A(\eta)Y-A(\eta)\mu\|_Y^2 + \mathbb{E}\,\underline{\|A(\eta)\mu-\mu\|}_Y^2.$$

$$= \frac{1}{n}\,\mathbb{E}\,(A(\eta)Y-A(\eta)\mu)^T(A(\eta)Y-A(\eta)\mu) + \|A(\eta)\mu-\mu\|_Y^2.$$

$$= (\qquad\qquad) + \frac{1}{n}\,\mu^T(A(\eta)-I)^T(A(\eta)-I)\,\mu.$$

$$= \underline{\frac{\sigma^2}{n}\,\text{trace}\,(A(\eta)^T A(\eta))} + \underline{\frac{1}{n}\,\mu^T(A(\eta)-I)^T(A(\eta)-I)\,\mu}.$$

let.

$$MSE(\eta) = \|(I-A(\eta))Y\|_Y^2. \qquad T m_\eta = A(\eta) Y.$$

$$GCV(\eta) = \frac{\|Y - T m_\eta\|_Y^2}{(1-\frac{1}{n}\operatorname{trace} A(\eta))^2}.$$

$$T m_\eta = A(\eta) Y$$

$$T_k m_\eta = T_k\left((\phi_1, \cdots \phi_q, \xi_1, \cdots \xi_n)(\mathcal{U}^T\mathcal{U} + \eta \mathcal{V})^{-1}\mathcal{U} Y\right).$$

$$= e_k^T \mathcal{U}(\mathcal{U}^T\mathcal{U} + \eta\mathcal{V})^{-1}\mathcal{U} Y$$

$$\underline{T m_\eta} = \begin{bmatrix} T_1 m_\eta \\ \vdots \\ T_n m_\eta \end{bmatrix} = . \quad \mathcal{U}(\mathcal{U}^T\mathcal{U} + \eta\mathcal{V})^{-1}\mathcal{U} Y = A(\eta) Y.$$

$$GCV(\eta) = \frac{\|Y - T m_\eta\|_Y^2}{(1-\frac{1}{n}\operatorname{trace} A(\eta))^2}. = \frac{\overset{\substack{MSE(\eta).\\ \uparrow}}{\|(I-A(\eta))Y\|_Y^2}}{(1-\frac{1}{n}\operatorname{trace} A(\eta))^2}$$

$$= \frac{MSE(\eta)}{(1-\frac{1}{n}\operatorname{trace} A(\eta))^2}.$$

$$\mathbb{E}(MSE(\eta)) = . \ \mathbb{E}\|(I-A(\eta))Y\|_Y^2 = \frac{1}{n}.\mathbb{E}\left(((I-A(\eta))Y)^T((I-A(\eta))Y)\right)$$

$$= \frac{1}{n}\mathbb{E}\left((I-A(\eta))(\mu+\varepsilon)\right)^T\left((I-A(\eta))(\mu+\varepsilon)\right)$$

$$= \frac{1}{n}.\mathbb{E}((I-A(\eta))\mu)^T((I-A(\eta))\mu)$$
$$+ \frac{2}{n}\mathbb{E}\left((I-A(\eta))\varepsilon\right)^T\left(\underline{(I-A(\eta))\mu}\right) \to 0.$$
$$+ \frac{1}{n}\mathbb{E}\left((I-A(\eta))\varepsilon\right)^T((I-A(\eta))\varepsilon).$$

$$=. \ \frac{1}{n}\mu^T(I-A(\eta))^T(I-A(\eta))\mu.$$
$$\frac{1}{n}\mathbb{E}\left(\sum_{i,j}\varepsilon_i\varepsilon_j \ e_i^T \underline{(I-A(\eta))^T(I-A(\eta))} e_j\right)$$

$$= \frac{1}{n}\mu^T(I-A(\eta))^T(I-A(\eta))\mu$$

$$+\frac{1}{n}\sigma^2 \ \text{trace}\,(I-A(\eta))^T(I-A(\eta)).$$

$$\text{trace}\,(I-A(\eta))^T(I-A(\eta)) = \text{trace}\,(I-2A(\eta)+A(\eta)^TA(\eta)).$$
$$= \text{trace}\, I - 2\,\text{trace}\, A(\eta) + \text{trace}\, A(\eta)^TA(\eta)$$
$$= n. - 2\,\text{trace}\, A(\eta) + \text{trace}\, A(\eta)^TA(\eta).$$

$$\mathbb{E}(MSE(\eta)) = \frac{1}{n}\mu^T(I-A(\eta))^T(I-A(\eta))\mu$$
$$+\frac{\sigma^2}{n}\big(- 2\,\text{trace}\, A(\eta) + \text{trace}\, A(\eta)^TA(\eta)\big) + \sigma^2.$$
$$\uparrow$$
$$=. \ \text{Risk}(\eta) + \sigma^2 - \frac{2\sigma^2}{n}\text{trace}\, A(\eta).$$

Theorem. (GCV theorem). let. $\tau_i(\eta) = \frac{1}{n}\text{trace}\,(A^i(\eta))$. $(i=1,2)$. let.

$$g(\eta) = \frac{2\tau_1(\eta) + \tau_1^2(\eta)/\tau_2(\eta).}{(1-\tau_1(\eta))^2.}$$

then. $$\frac{|\mathbb{E}\,GCV(\eta) - \sigma^2 - \text{Risk}(\eta)|}{\text{Risk}(\eta)} \leq g(\eta).$$

$$\uparrow$$

Proof.

$$\mathbb{E}\,GCV(\eta) - \sigma^2 - \text{Risk}(\eta) = \frac{\text{Risk}(\eta) + \sigma^2 - \frac{2\sigma^2}{n}\text{trace}\, A(\eta)}{(1-\frac{1}{n}\text{trace}\, A(\eta))^2} - \sigma^2 - \text{Risk}(\eta)$$

$$= \frac{\text{Risk}(\eta) + \sigma^2 - 2\sigma^2\tau_1(\eta)}{(1-\tau_1(\eta))^2} - \sigma^2 - \text{Risk}(\eta).$$

$$= \text{Risk}(\eta)\frac{1-(1-\tau_1(\eta))^2}{(1-\tau_1(\eta))^2.} + \sigma^2\cdot\frac{1-2\tau_1(\eta)-(1-\tau_1(\eta))^2}{(1-\tau_1(\eta))^2}$$

$$-\tau_1(\eta)^2.$$

$$= \text{Risk}(\eta).\frac{\tau_1(\eta)(2-\tau_1(\eta))}{(1-\tau_1(\eta))^2.} - \sigma^2\frac{\tau_1(\eta)^2}{(1-\tau_1(\eta))^2.}$$

$$\left|\frac{\mathbb{E}\,GCV(\eta) - \sigma^2 - \text{Risk}(\eta)}{\text{Risk}(\eta)}\right| \leq \left(\left|\frac{\tau_1(\eta)(2-\tau_1(\eta))}{(1-\tau_1(\eta))^2}\right| + \left|\frac{\sigma^2}{\text{Risk}(\eta)}\frac{\tau_1(\eta)^2}{(1-\tau_1(\eta))^2}\right|\right).$$

$$\frac{Risk(\eta)}{\sigma^2} = \frac{\frac{1}{n}\mu^T(I-A(\eta))^T(I-A(\eta))\mu + \frac{\sigma^2}{n}\cdot \mathrm{trace}\, A(\eta)^T A(\eta)}{\sigma^2 \qquad (I-A(\eta))\mu}$$

$$\geq T_2(\eta).$$

$$\left| \frac{\mathbb{E}G(V(\eta)) - \sigma^2 - Risk(\eta)}{Risk(\eta)} \right| \leq \frac{T_1(\eta)(2-T_1(\eta))}{(1-T_1(\eta))^2} + \frac{T_1(\eta)^2 / T_2(\eta)}{(1-T_1(\eta))^2} = g(\eta).$$

Spline.

Def. A spline of order $q \geq 1$, with knots at

$$0 < t_1 < \cdots < t_J < 1$$

is any function of form from.

$$g(t) = \sum_{i=0}^{q-1} \theta_i t^i + \sum_{j=1}^{J} \delta_j (t-t_j)_+^{q-1}$$

$\theta_0, \cdots \theta_{q-1}, \delta_q, \cdots \delta_J \in \mathbb{R}$.

$C\cdot(t-t_j)_+$

- $\forall j$. g is polynomial of order q on any interval $[t_j, t_{j+1})$.
- for $q \geq 2$, g has $q-2$ continuous derivative.
- the $(q-1)^{st}$ derivative of g is a step function with jump at $t_1 \cdots t_J$

Given q, J, knots $0<t_1<\cdots<t_J<1$. consider the vector space. $S^q(t_1, \cdots t_J)$ of spline of order q.

$$B = \{1, t, \cdots t^{q-1}, (t-t_1)_+^{q-1}, \cdots (t-t_J)_+^{q-1}\}.$$

thus. $\mathrm{span}\, B. = S^q(t_1, \cdots t_J). \Rightarrow \dim S^q(t_1, \cdots t_J) = q + J$

$\boxed{\text{B-spline}}$

define $2(q-1)$ additional knots.

$q-1$

$$t_{-(q-1)} < \cdots < t_{-1} < \underset{t_0}{0} < t_1 < \cdots < t_J < \underset{t_{J+1}}{1} < J_{J+2} < \cdots < t_{J+q}.$$

$q-1$.

$\geq (q-1)$.

Takes $0 = t_{-(q-1)} = \cdots = t_{-1}$ $\quad t_{J+2} = \cdots = t_{J+q} = 1$.

<u>Def</u>. q-th. order divided difference of a function g at $\{t_i, \cdots t_{i+q}\}$. is.

$$[t_i, \cdots t_{i+q}]g = \frac{[t_{i+1}, \cdots t_{i+q}]g - [t_i, \cdots t_{i+q-1}]g}{t_{i+q} - t_i}.$$

with $[t_i]g = g(t_i)$

Example: $$[t_i, t_{i+1}]g = \frac{[t_{i+1}]g - [t_i]g}{t_{i+1} - t_i} = \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i}.$$

$$[t_i, t_{i+1}, t_{i+2}]g = \frac{[t_{i+1}, t_{i+2}]g - [t_i, t_{i+1}]g}{t_{i+2} - t_i}.$$

$$= \frac{1}{t_{i+2} - t_i} \cdot \frac{g(t_{i+2}) - g(t_{i+1})}{t_{i+2} - t_{i+1}} + \frac{1}{t_{i+2} - t_i} \cdot \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i}.$$

$$= (\ )g(t_{i+2}) + (\quad)g(t_{i+1}) + (\quad)g(t_i).$$

$$\underline{[t_i, \cdots t_{i+q}]g} = \sum_{k=0}^{q} \alpha_k \underline{g(t_{k+i})}$$

Theorem. let. $$P_q = \sum_{i=1}^{q} g(t_i)\, l_i(x).$$

$$l_i(x) = \prod_{\substack{1 \leq j \leq q \\ j \neq i}} \frac{x - t_j}{t_i - t_j}.$$

Then. ①. P_q. is the unique q^{th}. order polynomial that agree g at t_i $(1 \leq i \leq q)$.

②. $\forall q \in \mathbb{Z}_{>0}$, the coefficient of x^q. in P_{q+1} is.

$[t_1, \cdots t_{1+q}]g$

<u>Proof</u>.

①. uniqueness. Suppose $P(x) = \sum_{i=0}^{q-1} a_i x^i$ Satisfying.

$P(t_i) = g(t_i)$. $\quad 1 \leq i \leq q \quad (*)$

$$\begin{bmatrix} 1 & t_1, & \cdots & t_1^{q-1} \\ 1 & t_2, & \cdots & t_2^{q-1} \\ \vdots & \vdots & & \vdots \\ 1 & t_q & & t_q^{q-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_q) \end{bmatrix}.$$

$$A \quad x \quad = \quad y.$$

$\det A \neq 0. \implies x = A^{-1} y.$

$$P_q(t_i) = \sum_{i=1}^{q} g(t_i) \prod_{\substack{1\le j\le q \\ j\neq i}} \frac{x-t_j}{t_i-t_j}. \qquad l_i.$$

$$l_i(t_i) = \prod_{\substack{1\le j\le q \\ j\neq i}} \frac{t_i-t_j}{t_i-t_j} = 1 \qquad l_i(t_k). = \frac{t_k-t_k}{t_i-t_k} \prod_{\substack{j\neq i \\ j\neq k}} (\quad) = 0.$$

$$l_i(t_j) = \delta_{ij}.$$

$$P_q(t_i) = g(t_i).$$

②. if $q=1$, $\quad P_q(x) = g(t_1). \dfrac{x-t_2}{t_1-t_2} + g(t_2)\dfrac{x-t_1}{t_2-t_1}$

coefficient of x^1 :

$$\frac{g(t_1)}{t_1-t_2} + \frac{g(t_2)}{t_2-t_1} = \frac{g(t_2)-g(t_1)}{t_2-t_1} = [t_1, t_2]g.$$

$\tilde{P}_q$

$t_1, t_2, \cdots, t_q, t_{q+1}$

$P_q.$

$$P_q(x) = \sum_{1\le i\le q} g(t_i) \prod_{\substack{1\le j\le q \\ j\neq i}} \frac{x-t_j}{t_i-t_j}.$$

$$\tilde{P}_q(x) = \sum_{2\le i\le q+1} g(t_i) \prod_{\substack{2\le j\le q+1 \\ j\neq i}} \frac{x-t_j}{t_i-t_j}$$

$P_q(t_1) = g(t_1).$ $\qquad \tilde{P}_q(t_1) = (\quad).$

$P_q(t_j) = g(t_j) \quad 2\le j\le q \qquad \tilde{P}_q(t_j) = g(t_j)$

$P_q(t_{q+1}) = (\qquad).$ $\qquad \tilde{P}_q(t_{q+1}) = g(t_{q+1}).$

$$\boxed{P_{q+1}(x) = f(x)P_q(x) + g(x)\tilde{P}_q(x) \qquad P_{q+1}(t_j) = g(t_j) \quad 1\le j\le q+1}$$

$$P(x). = (ax+b)P_q(x) + (cx+d)\tilde{P}_q(x).$$

$$P(t_1) = \underline{(at_1+b)}\, P_q(t_1) + \underline{(ct_1+d)}\, \tilde{P}_q(t_1). = g(t_1).$$

$$P(t_{q+1}). = (at_{q+1}+b)P_q(t_{q+1}) + (ct_{q+1}+d)\tilde{P}_q(t_{q+1}). = g(t_{q+1}).$$

$$(at_1+b)=1 \qquad (ct_1+d)=0$$

$$(at_{q+1}+b)=0 \qquad (ct_{q+1}+d)=1.$$

$$a=\frac{1}{t_1-t_{q+1}} \qquad b=\frac{-t_1}{t_{1+q}-t_1}$$

$$c=\frac{1}{t_1-t_{q+1}} \qquad d=\frac{t_{1+q}}{t_{1+q}-t_1}.$$

$$P(x)=\frac{x-t_{1+q}}{t_1-t_{1+q}}P_q(x)+\frac{x-t_1}{t_{1+q}-t_1}\tilde{P}_q(x). \qquad t_j$$

$$P(t_1)=P_q(t_1)=g(t_1).$$

$$P(t_{q+1})=\tilde{P}_q(t_{q+1})=g(t_{q+1}).$$

$2\le j\le q$

$$P(t_j)=\underbrace{\frac{t_j-t_{1+q}}{t_1-t_{1+q}}}\,g(t_j)+\underbrace{\frac{t_j-t_1}{t_{1+q}-t_1}}\,g(t_j).=g(t_j).$$

i.e. $1\le j\le q+1$

$$P(t_j)=g(t_j).$$

By uniqueness.

$$P_{q+1}(x)=P(x)=\frac{x-t_{1+q}}{t_1-t_{1+q}}P_q(x)+\frac{x-t_1}{t_{1+q}-t_1}\tilde{P}_q(x).$$

the coefficient of x^q:

$$\frac{1}{t_1-t_{1+q}}[t_1,\cdots t_q]g+\frac{1}{t_{1+q}-t_1}[t_2,\cdots t_{q+1}]g$$

$$=\frac{[t_2\cdots t_{q+1}]g-[t_1,\cdots t_q]g}{t_{1+q}-t_1}=[t_1,\cdots t_{q+1}]g.$$

Corollary. If g is a polynomial of order of q on $[t_i,t_{i+q}]$ then.

$$[t_i,\cdots t_{i+q}]g=0.$$

Proof. Let.

$$P_q(x)=\sum_{0\le k\le q}g(t_{i+k})\underline{\prod_{\substack{0\le j\le q\\ j\ne k}}\left(\frac{x-t_{i+j}}{t_{i+k}-t_{i+k}}\right)} \qquad x^{q-1}$$

$\underline{[t_i,\cdots t_{i+q}]g}$ is the coefficient of $\underline{x^q}$

$$g(t_j)=g(t_j)\quad\forall j=i,\cdots i+q.$$

$$\Rightarrow.\quad g(x)=P_q(x).$$

$\Rightarrow. \quad 0 = [t_i, \cdots t_{i+q}] g.$ $\square$

Denote by $\alpha_{rq}^{[i]}$.

$$(t_{i+q}-t_i)[t_i, \cdots t_{i+q}] g = \sum_{r=0}^{q} \alpha_{rq}^{[i]} g(t_{i+r}). \quad \star$$

let $g(x) = (x-t)_+^{q-1} = \begin{cases} (x-t)^{q-1} & x \geq t \\ 0 & x < t \end{cases}$

$$N_{iq}(t) = (t_{i+q}-t_i)[t_i, \cdots t_{i+q}] g$$
$$= \sum_{r=0}^{q} \alpha_{rq}^{[i]} (t_{i+r}-t)_+^{q-1}.$$

These function are called B- spline.

Theorem. $\{N_{iq}(\cdot)\}_{i=-(q-1)}^{J}$ satisfy

① . N_{iq} is a polynomial of order q on each interval $[t_i, t_{i+1})$

② . $N_{iq}(t) = 0. \quad \forall t \notin [t_i, t_{i+q}]$.

③ Let $N_{i1} = I_{[t_i, t_{i+1}]}(t)$ then.

$$N_{iq}(t) = \frac{t-t_i}{t_{i+q-1}-t_i} N_{i(q-1)}(t) + \frac{t_{i+q}-t}{t_{i+q}-t_{i+1}} N_{(i+1)(q-1)}(t).$$

Proof. $\forall t > t_{i+q}. \quad (t_{i+r}-t)_+^{q-1} = 0. \quad (0 \leq r \leq q).$

$(t > t_{i+q} > t_{i+r} \Rightarrow t_{i+r}-t < 0).$

$\forall t < t_i \quad t_{i+r} > t_i > t \Rightarrow t_{i+r}-t > 0.$

$$\Rightarrow (t_{i+r}-t)_+^{q-1} = (t_{i+r}-t)^{q-1}. \quad 0 \leq r \leq q.$$

$$\Rightarrow. N_{iq}(t) = \sum_{r=0}^{q} \alpha_{rq}^{[i]} (t_{i+r}-t)_+^{q-1}.$$

$$= \sum_{r=0}^{q} \alpha_{rq}^{[i]} (t_{i+r}-t)^{q-1} \quad t < t_i.$$

$$= (t_{i+q}-t_i)[t_i, \cdots t_{i+q}] g \quad t < t_i$$

$$\underline{g(x) \stackrel{\triangle}{=} (x-t)^{q-1}.}$$

$$= 0.$$

$\Rightarrow. \quad N_{iq}(t) = 0 \quad \forall t \notin [t_i, t_{i+q}].$

Given $\{t_i\}_{i=1}^{J}$

$$\left.\begin{array}{l} supp(N_{iq}) \subset [t_i, t_{i+q}]. \\ supp(N_{jq}) \subset [t_j, t_{j+q}] \end{array}\right\} \Rightarrow \text{ if } |i-j| > q-1, \text{ then } supp(N_{iq}) \cap supp(N_{jq}) = \phi.$$

$$\Rightarrow \langle N_{jq}, N_{iq} \rangle_2 = \int_0^1 N_{iq}(t) N_{jq}(t) dt = 0 \quad \text{if } |i-j| > q-1.$$

"nearly orthogonal."

providing a basis for $S^q(t_1, \cdots t_J)$

Def. A spline g of order $2q$ $(q \geq 1)$ with knots at.

$$0 < t_1 < \cdots < t_J < 1$$

is said to be a natural spline of order $2q$ if g is a polynomial of order q outside of $[t_1, t_J]$. $N^{2q}[t_1, \cdots t_J)$.

i.e. $\displaystyle g(t) = \sum_{i=0}^{2q-1} \theta_i t^i + \sum_{j=1}^{J} \delta_j (t-t_j)_+^{2q-1}$. $\forall t$.

$\star$.. $\displaystyle g(t) = \sum_{n=0}^{q-1} \alpha_n t^n$ $\quad t \notin [t_1, t_q]$.

Remark.

$\forall g \in N^{2q}(t_1, \cdots t_J)$.

$$\boxed{g^{(j)}(0) = g^{(j)}(1) = 0} \; \star. \qquad j = q, \cdots 2q-1.$$

$1, 0 \notin [t_1, t_q]$.

It follows that.

$$\dim N^{2q}(t_1, \cdots t_J).$$
$$= \dim \left(S^q(t_1, \cdots t_J)\right) - 2q = 2q + J - 2q = J$$

Theorem: Given $0 < t_1, \cdots < t_J < 1$, and $\{Z_j\}_{j=1}^{J}$, $\exists! g \in N^{2q}(t_1, \cdots t_J)$ such that.

$$g(t_j) = Z_j$$

natural interpolating spline

Theorem. For knots. $0<t_1<\cdots<t_J<1$, let $\tilde{g} \in N^{\chi}(t_1, \ldots t_J)$ that interpolate $z_1, \ldots z_J$. Then $\forall g \in W_q[0,1]$. that interpolate $\{z_j\}$.

$$\int_0^1 |\tilde{g}^{(q)}(t)|^2 dt \leq \int_0^1 |g^{(q)}(t)|^2 dt.$$

"=" holds $\Leftrightarrow$ $g = \tilde{g} + h$. for h a q^{th} polynomial, $h(t_j)=0$.
If $J \geq q$, $h=0$ on $[0,1]$.

Proof.

Let $g \in W_q[0,1]$. and

$$g(t_j) = z_j.$$

Set $h = g - \tilde{g}$, Then

$$\boxed{h(t_j) = g(t_j) - \tilde{g}(t_j) = z_j - z_j = 0 \quad \forall\ 1 \leq j \leq J}$$

$$\int_0^1 \tilde{g}^{(q)}(t) h^{(q)}(t) dt. = -\int_0^1 \tilde{g}^{(q+1)}(t) h^{(q-1)}(t) dt + \tilde{g}^{(q)} h^{(q-1)}(t) \Big|_0^1.$$

$$= -\int_0^1 \tilde{g}^{(q+1)}(t) h^{(q-1)}(t) dt.$$

$$= (-1)^{q-1} \int_0^1 \tilde{g}^{(2q-1)}(t) h'(t) dt.$$

$$= (-1)^{q-1} \sum_{j=1}^{J-1} \int_{t_j}^{t_{j+1}} \tilde{g}^{(2q-1)}(t) h'(t) dt.$$

$$= (-1)^{q-1} \sum_{j=1}^{J-1} \Big(\underbrace{\tilde{g}^{(2q-1)}(t_{j+1}^-) h(t_{j+1}) - \tilde{g}^{(2q-1)}(t_j) h(t_j)}_{\tilde{g}^{(2q-1)}(t_j^+)} \Big).$$

$$= (-1)^{q-1} \sum_{j=1}^{J-1} \tilde{g}^{(2q-1)}(t_j^+) \big(h(t_{j+1}) - h(t_j) \big).$$

$$= 0.$$

$$\int_0^1 |g^{(q)}(t)|^2 dt = \int_0^1 \big(\tilde{g}^{(q)}(t)^2 + 2 \underline{h^{(q)}(t) \tilde{g}^{(q)}(t)} + h^{(q)}(t)^2 \big) dt.$$

$$= \int_0^1 |\tilde{g}^{(q)}(t)|^2 dt + \int_0^1 |h^{(q)}(t)|^2 dt.$$

$$\geq \int_0^1 |\tilde{g}^{(q)}(t)|^2 dt.$$

"=" holds $\iff \int_0^1 |h^{(q)}(t)|^2 dt = 0.$

$\iff h^{(q)}(t) = 0 \quad a.e.$

$g, \tilde{g} \in W_q[0,1] \Rightarrow. \quad h = g - \tilde{g} \in W_q[0,1].$

$$h(t) = \sum_{i=1}^{q-1} \frac{h^{(i)}(0)}{i!} t^i + \int_0^1 \frac{(t-u)_+^{q-1}}{(q-1)!} h^{(q)}(u)\, du.$$

$h^{(q)}(t) = 0 \ a.e. \iff \underline{h(t) = \sum_{i=1}^{q-1} \frac{h^{(i)}(0)}{i!} t^i}.$

$h(t_j) = 0 \quad 1 \leq j \leq J.$

if $J \geq q. \quad h(x) = 0. \quad \forall \ x \in [0,1].$