THM 1.2.1

Proof (1). 记

$$\mathcal{G}_1 = \{A : A \in m(\mathcal{C}),\ A^c \in m(\mathcal{C}),\quad A \cap B \in m(\mathcal{C}) \text{ for all } B \in \mathcal{C}\}.$$

$$\mathcal{G}_2 = \{A : A \in m(\mathcal{C}),\ A^c \in m(\mathcal{C}).\quad A \cap B \in m(\mathcal{C}) \text{ for all } B \in \mathcal{C}\}$$

Claim ①. $\mathcal{G}_1 \supset \mathcal{C}$ and $\mathcal{G}_1$ is monotone class and thus. $\mathcal{G}_1 = m(\mathcal{C})$

② $\mathcal{G}_2 \supset \mathcal{C}$ and $\mathcal{G}_2$ is monotone class and thus $\mathcal{G}_2 = m(\mathcal{C})$

Proof of ①:

- 给定 $A \in \mathcal{C}$，则
  - $A \in \mathcal{C} \Rightarrow A \in \mathcal{C} \subset m(\mathcal{C})$
  - $A^c \in \mathcal{C} \Rightarrow A^c \in \mathcal{C} \subset m(\mathcal{C})$
  - $A \cap B \in \mathcal{C}$ for all $B \in \mathcal{C} \Rightarrow A \cap B \in m(\mathcal{C})$ for all $B \in \mathcal{C}$

  $\Rightarrow A \in \mathcal{G}$.

  THM 1.2.2

  故，$\mathcal{C} \subset \mathcal{G}_1$
- 令 $\{A_n\}_{n=1}^{\infty} \subset \mathcal{G}_1$，$A_n \uparrow A$ ($A_n \downarrow A$ 同理) 则 $\forall n \geq 1$

  $A_n \in m(\mathcal{C}) \Rightarrow A = \lim A_n \in m(\mathcal{C})$

  $A_n^c \in m(\mathcal{C}) \Rightarrow A^c = \lim A_n^c \in m(\mathcal{C})$

  $A_n \cap B \in m(\mathcal{C})$ for all $B \in \mathcal{C} \Rightarrow A \cap B = (\bigcup A_n) \cap B = \bigcup (A_n \cap B) \in m(\mathcal{C})$

这说明：$\mathcal{G}_1$ 单调类.

Proof of ②.

令 $A \in \mathcal{C}$.

$A \in \mathcal{C} \Rightarrow A \in m(\mathcal{C})$

$A^c \in \mathcal{C} \Rightarrow A^c \in m(\mathcal{C})$.

$\forall B \in m(\mathcal{C}) = \mathcal{G}_1$, $A \cap B \in m(\mathcal{C})$ (def of $\mathcal{G}_1$) $\Rightarrow A \cap B \in m(\mathcal{C})$.

$\Rightarrow \mathcal{C} \subset \mathcal{G}_2$

$\mathcal{G}_2$ 为单调集易证.

从而 $m(\mathcal{C}) = \mathcal{G}_2$

$A \in m(\mathcal{C}) \Rightarrow A^c \in m(\mathcal{C})$, $A, B \in m(\mathcal{C}) \Rightarrow A \cap B \in m(\mathcal{C})$.

故 $m(\mathcal{C})$ 为代数.

algebra + monotone class $\Rightarrow$ σ-algebra.

从而 $m(\mathcal{C}) \supset \sigma(\mathcal{C})$。 $m(\mathcal{C}) \subset \sigma(\mathcal{C})$ 显然. $\square$

THM 1.2.4.

Proof 小记.

$\mathcal{D} = \{A : A^c \in \mathcal{C}\}$. 则 $\mathcal{C} = \{A : A^c \in \mathcal{D}\}$.

记. $\mathcal{G} = \{A : A^c \in m(\mathcal{C})\}$ $\mathcal{H} = \{A : A^c \in m(\mathcal{D})\}$.

Claim $\mathcal{D} \subset \mathcal{G}$, $\mathcal{G}$ monotone class and thus $m(\mathcal{D}) \subset \mathcal{G}$

$\mathcal{C} \subset \mathcal{H}$, $\mathcal{H}$ monotone class and thus $m(\mathcal{C}) \subset \mathcal{H}$.

$A \in \mathcal{D} \Leftrightarrow A^c \in \mathcal{C} \subset m(\mathcal{C}) \Rightarrow A^c \in m(\mathcal{C}) \Rightarrow A \in \mathcal{G}$.

从而 $\mathcal{D} \subset \mathcal{G}$

令 $\{A_i\} \subset \mathcal{G}$, $A_i \uparrow A$, 则

$$\{A_i^c\} \subset m(\mathcal{C}), \quad A_i^c \downarrow A^c.$$

$$\Rightarrow \quad A^c = \lim A_i^c \in m(\mathcal{C}) \Rightarrow A \in \mathcal{G}$$

故 $\mathcal{G}$ 为单调集. 从而 $m(\mathcal{D}) \subset \mathcal{G}$, 这说明

$$A \in m(\mathcal{D}) \Rightarrow A \in \mathcal{G} \Rightarrow A^c \in m(\mathcal{C}). \qquad (*)$$

同理 $m(\mathcal{C}) \subset \mathcal{H}$, 这说明

$$A \in m(\mathcal{C}) \Rightarrow A \in \mathcal{H} \Rightarrow A^c \in m(\mathcal{D}) \qquad (**)$$

结合 (*), (**) 有

$$m(\mathcal{D}) = \{A : A^c \in m(\mathcal{C})\}. \text{反} \quad m(\mathcal{C}) = \{A : A^c \in m(\mathcal{D})\}.$$

Claim: $m(\mathcal{D}) = \sigma(\mathcal{D})$.

由题干知:

- $A \in m(\mathcal{D}) \Rightarrow A^c \in m(\mathcal{C}) \Rightarrow A \in m(\mathcal{C}) \Rightarrow A^c \in m(\mathcal{D})$.
- $A, B \in m(\mathcal{D}) \Rightarrow A^c, B^c \in m(\mathcal{C}) \Rightarrow A^c \cup B^c \in m(\mathcal{C}) \Rightarrow (A \cap B)^c \in m(\mathcal{C}) \Rightarrow A \cap B \in m(\mathcal{D})$

上两条表明 $m(\mathcal{D})$ 符合 THM 1.2.3 条件, 即

$$m(\mathcal{D}) = \sigma(\mathcal{D}).$$

从而

- $\sigma(\mathcal{D}) = m(\mathcal{D}) = m(\mathcal{C}) \subset \sigma(\mathcal{C}) \Rightarrow \sigma(\mathcal{D}) \subset \sigma(\mathcal{C})$.
- $\mathcal{C} \subset m(\mathcal{C}) = m(\mathcal{D}) = \sigma(\mathcal{D}) \Rightarrow \mathcal{C} \subset \sigma(\mathcal{D}) \Rightarrow \sigma(\mathcal{C}) \subset \sigma(\mathcal{D})$

即 $\sigma(\mathcal{C}) = \sigma(\mathcal{D}) = m(\mathcal{C})$.

定义1.2.8 的结论.

(1). 只需证若 $A(\omega)\cap A(\omega')\neq\phi$, 则 $A(\omega)=A(\omega')$.

断言 $\omega\in A(\omega')$, 若不然:

$$\omega\in A(\omega')^c=\bigcup_{B\in\mathcal{F}_{\omega'}} B^c$$

故存在 $B_0\in\mathcal{F}_{\omega'}$ 使得 $\omega\in B_0^c$

但 $\mathcal{F}$ 为 σ-域, 故 $B_0^c\in\mathcal{F}$. 故 $B_0^c\in\mathcal{F}_\omega$ 从而

$$A(\omega)=\bigcap_{B\in\mathcal{F}_\omega} B\subset B_0^c$$

但 $$A(\omega')=\bigcap_{B\in\mathcal{F}_{\omega'}} B\subset B_0$$

从而 $$A(\omega)\cap A(\omega')=\phi.$$

矛盾! 故 $\omega\in A(\omega')$. 从而 $\forall B\in\mathcal{F}_{\omega'}$

$$\omega\in B \qquad B\in\mathcal{F}.$$

从而 这样的B是 $\mathcal{F}_\omega$ 中的元素, 即.

$$\mathcal{F}_{\omega'}\subset\mathcal{F}_\omega$$

故 $$A(\omega')=\bigcap_{B\in\mathcal{F}_{\omega'}} B\supset\bigcap_{B\in\mathcal{F}_\omega} B=A(\omega)$$

同理 $$A(\omega')\subset A(\omega)$$

即 $$A(\omega)=A(\omega')$$

(2). 显然有 $\mathcal{F}_\omega\supset\mathcal{C}_\omega$ 故

$$A(\omega)=\bigcap_{B\in\mathcal{F}_\omega} B\subset\bigcap_{B\in\mathcal{C}_\omega} B$$

记 $\Omega_0=\bigcap_{B\in\mathcal{C}_\omega} B$

$\mathcal{F}_0=\mathcal{F}|\Omega_0=\{B\cap\Omega_0 ; B\in\mathcal{F}\}$. ~ σ-field.

由习题 1.2.1 可知

$$\mathcal{F}_0=\sigma(\mathcal{C}\cap\Omega_0). \qquad \mathcal{C}\cap\Omega_0 \text{ 为一个代数}$$

Claim: $\mathcal{C}\cap\Omega_0=\{\phi,\Omega_0\}$, 则 $\mathcal{F}_0=\{\phi,\Omega_0\}$.

事实上: 令 $A\cap\Omega_0\in\mathcal{C}\cap\Omega_0$, $(A\in\mathcal{C})$

①.若 $\omega\in A$, 则 $A\cap\Omega_0=\Omega_0$. (由 Ω_0 构造即得)

②. 若 $\omega \notin A$，则 $\omega \in A^c \Rightarrow A^c \cap \Omega_0 = \Omega_0 \Rightarrow A \cap \Omega_0 = \phi$

从而 $\mathcal{F}_0 = \{\phi, \Omega_0\}$

. 但由 $A(\omega) \subset \Omega_0$ 知 $A(\omega)$ 可看作 $\mathcal{F}_0$ 中的原子：

$$A(\omega) \subset \Omega_0 \Rightarrow A(\omega) = \bigcap_{B \in \mathcal{F}_\omega} B = \left(\bigcap_{B \in \mathcal{F}_\omega} B\right) \cap \Omega_0 = \bigcap_{B \in \mathcal{F}_\omega} (B \cap \Omega_0)$$

而 $B \in \mathcal{F}_\omega \Leftrightarrow B \cap \Omega_0 \in \mathcal{F}_\omega \cap \Omega_0$.

故 $A(\omega) = \bigcap_{B \cap \Omega_0 \in \mathcal{F}_\omega \cap \Omega_0} (B \cap \Omega_0) = \bigcap_{\tilde{B} \in \mathcal{F}_\omega \cap \Omega_0} \tilde{B}$

但
$$\begin{aligned}
\mathcal{F}_\omega \cap \Omega_0 &= \{B \cap \Omega_0 : B \in \mathcal{F}, \omega \in B\} \\
&= \{B \cap \Omega_0 \in \mathcal{F}_0 : \omega \in B\} \\
&= \{B \cap \Omega_0 \in \mathcal{F}_0 : \omega \in B \cap \Omega_0\} \\
&= \{\tilde{B} \in \mathcal{F}_0 : \omega \in \tilde{B}\} = (\mathcal{F}_0)_\omega .
\end{aligned}$$

从而 $A(\omega) = \bigcap_{B \in \mathcal{F}_\omega \cap \Omega_0} B = \bigcap_{B \in (\mathcal{F}_0)_\omega} B$

即 $A(\omega)$ 为 $\mathcal{F}_0$ 中的原子

由 $A(\omega) \neq \phi$ $(\omega \in A(\omega))$ 知 $A(\omega) = \Omega_0$.