Signed measure.

Def. Let $(X, \mathcal{M})$ be a measurable space. A sign measure on $(X, \mathcal{M})$ is a function $\nu: \mathcal{M} \to [-\infty, +\infty]$ such that

① $\nu(\phi) = 0$

②. ν assumes ~~that~~ at most one of the value $\pm\infty$.

③. $\{E_j\} \subset \mathcal{M}$, $E_i \cap E_j = \phi$ $(i \neq j)$ then

$$\nu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \sum \nu(E_j)$$

if $\nu\left(\bigsqcup_{j=1}^{\infty} E_j\right) < \infty$, then $\sum \nu(E_j)$ converges absolutely.

Remark: We could call measure and positive measure.

Example ① μ_1, μ_2 measure on $(X, \mathcal{M})$, one of which is finite, then

$\nu = \mu_1 - \mu_2$ is a signed measure.

②. $f: \mathcal{M} \to [-\infty, +\infty]$ measurable,

$$\int f^+ d\mu < \infty \quad \text{or} \quad \int f^- d\mu < \infty$$

then $\nu(E) = \int_E f \, d\mu$ signed measure.

Proposition. Let ν be a signed measure on $(X, \mathcal{M})$. If $\{E_j\}$ increas in $\mathcal{M}$,

$$\nu\left(\bigcup_1^{\infty} E_j\right) = \lim_{j \to \infty} \nu(E_j)$$

If $\{E_j\}$ decrease $\mathcal{M}$, $|\nu(E_1)| < \infty$,

$$\nu\left(\bigcap_{j=1}^{\infty}\right) = \lim_{j \to \infty} \nu(E_j)$$

Proof. Let $F_j = \bigcup_{i=1}^{j} E_i \setminus F_{j-1}$, $F_1 = E_1$, then

$$\nu\left(\bigcup E_j\right) = \nu\left(\bigsqcup F_j\right) = \sum \nu(F_j) \qquad \text{if } |\nu(\bigcup E_j)| < \infty$$

$$= \nu(E_n) + \sum_{j=n+1}^{\infty} \nu(F_j)$$

$\nu(\bigcup E_j) = \infty$ is obvious.

Theorem (Jordan-Hahn decomposition theorem) Let ν be a signed measure on $(\Omega, \mathcal{F})$

Let
$$\nu^+(A) = \sup\{\nu(B) : B \subset A, B \in \mathcal{F}\} \qquad \forall A \in \mathcal{F}.$$
$$\nu^-(A) = \sup\{-\nu(B) : B \subset A, B \in \mathcal{F}\}$$

then: • ν^+, ν^- are positive measure. one of which is finite. ~~and~~

• $\nu = \nu^+ - \nu^-$

• $\exists\, D \in \mathcal{F}$ such that
$$\nu^+(A) = \nu(A \cap D), \quad \nu^-(A) = -\nu(A \cap D^c)$$

Remark: • $\nu = \underset{\text{positive}}{\nu^+} - \underset{\text{negative variation.}}{\nu^-} \rightsquigarrow$ Jordan decomposition.

positive/negative variation.

$\Omega = D \cup D^c \rightsquigarrow$ Hahn decomposition.

• Let $|\nu| = \nu^+ + \nu^- \rightsquigarrow$ total variation

$\|\nu\|_{TV} = |\nu|(\Omega)$.

We say ν σ-finite if $|\nu|$ σ-finite.

• Let $h = I_D - I_{D^c}$, then,
$$d\nu = h\, d|\nu|, \quad |\nu| = h\, d\nu.$$

<u>Proof of Jordan-Hahn</u>

Assume, WLOG, that $\nu(A) > -\infty$, $\forall A \in \mathcal{F}$ ~~be~~

Claim ①: $\exists\, D \in \mathcal{F}$ such that
$$A \in \mathcal{F} \begin{cases} A \subset D \Rightarrow \nu(A) \geq 0 \\ A \subset D^c \Rightarrow \nu(A) \leq 0. \end{cases}$$

<u>Proof</u> of claim ①: Let $\mathcal{B} = \{B \in \mathcal{F} : \nu^+(B) = 0\}$
$$= \{B \in \mathcal{F} : C \subset B, C \in \mathcal{F} \Rightarrow \nu(C) \leq 0\}.$$

then • $\mathcal{B}$ closed under countable union

• $B \in \mathcal{B}$, $G \in \mathcal{F}$, $G \subset B \Rightarrow G \in \mathcal{B}$.

Let $\lim_n \nu(B_n) = \inf\{\nu(B) : B \in \mathcal{B}\} =: \beta$

where $B_n \in \mathcal{B}$, then $\hat{B} =: \bigcup_n B_n \in \mathcal{B}$. ①

$$\beta \leq \nu(\bigcup B_n) = \nu(B_m) + \nu(\bigcup_n B_n \setminus B_m)$$
$$\leq \nu(B_m), \quad m \geq 1$$

It follows that $\beta = \nu(\bigcup B_n)$. Let $D = (\bigcup B_n)^c$, then $D^c \in \mathcal{B}$.
By the definition of $\mathcal{B}$. $\forall A \in \mathcal{F}$, $A \subset D^c$, we have

$$\nu(A) \leq 0.$$

Claim ②: $[A \in \mathcal{F}, A \subset D, \nu(A) < 0 \Rightarrow \nu^+(A) > 0.]$

In fact: assume $\nu^+(A) = 0$, then $A \in \mathcal{B}$ then. $D^c \cup A \in \mathcal{B}$, It follows that

$$\beta = \nu(D^c) > \nu(D^c) + \nu(A) = \nu(D^c \cup A)$$

contradiction!

By the definition of ν^+, let $A \subset D$, $\nu(A) < 0$, then $\nu^+(A) > 0$, then $\exists A_1 \in \mathcal{F}$, $A_1 \subset A$, such that

$$\nu(A_1) > \frac{1}{2}\nu^+(A) > 0.$$

then. $\left.\begin{cases} A \subset D \Rightarrow \quad A \setminus A_1 \subset D, \\ \nu(A \setminus A_1) = \nu(A) - \nu(A_1) < 0. \\ A \setminus A_1 \in \mathcal{F}. \end{cases}\right\} \xrightarrow{\text{Claim ②}} \nu^+(A \setminus A_1) > 0$

By induction, we can find $\{A_n\} \subset \mathcal{F}$, with.

$$\begin{cases} A_n \in \mathcal{F}, A_n \subset D, A_n \subset A \setminus \sum_{k=1}^{n-1} A_k. \\ \nu(A_n) > \frac{1}{2}\nu^+(A \setminus \sum_{k=1}^{n-1} A_k) > 0. \quad (*) \end{cases}$$

It follows that $\nu(A) = \nu(A \setminus \sum_1^{\infty} A_k) + \sum_1^{\infty} \nu(A_k)$

Since $\nu(A) < 0$, $\nu(A_k) > 0$, we have

$$\sum \nu(A_k) < \infty \Rightarrow \nu(A_k) \to 0.$$

By (*) $\nu^+(A \setminus \sum_{k=1}^{\infty} A_k) = 0.$

But

$$\left.\begin{array}{l} A \setminus \sum_{k=1}^{\infty} A_k \in \mathcal{F} \\ A \setminus \sum_{k=1}^{\infty} A_k \subset D. \\ \nu^+(A \setminus \sum_{k=1}^{\infty} A_k) = 0 \end{array}\right\} \xrightarrow{\text{Claim ②}} \nu(A \setminus \sum A_k) \geq 0.$$

$$\Downarrow$$

$$\nu(A) = \nu(A \setminus \sum A_k) + \sum \nu(A_k) > 0.$$

Contradiction!

Let $A \in \mathcal{F}$. $B \in \mathcal{F}$, $B \subset A$, then

$$\nu(B) + \nu((A \setminus B) \cap D) = \nu((A \cap D) \cup B)$$
$$= \nu(A \cap D) + \nu(B \cap D^c)$$

$$\Rightarrow \quad \nu(B) \leq \nu(A \cap D)$$

But

$$\left\{\begin{array}{l} \nu^+(A) = \sup\{\nu(B) : B \subset A,\ B \in \mathcal{F}\} \leq \nu(A \cap D). \\ \nu(A \cap D) \leq \nu^+(A) \text{ is obvious.} \end{array}\right.$$

$$\Downarrow$$

$$\nu(A \cap D) = \nu^+(A).$$

Definition ① ν: signed measure, μ: positive measure on $(\Omega, \mathcal{F})$.

$$\nu \ll \mu \iff \mu(E) = 0 \Rightarrow \nu(E) = 0.$$

We say that: ν absolutely continuous with respect to μ

②. ν, μ: Signed measure. $\nu \perp \mu$ if $\exists N$ such that

$$\left\{\begin{array}{l} \nu(M) = 0 \ \forall M \subset N \\ \mu(M) = 0 \ \forall M \subset N^c. \end{array}\right.$$

We say ν and μ is mutual singularity

Remark: By Jordan-Hahn: for ν signed measure. $\exists \nu^+, \nu^-$ positive measure such that

$$\nu = \nu^+ - \nu^- \qquad \nu^+ \perp \nu^-.$$

Theorem. ν finite signed measure μ positive measure on $(\Omega, \mathcal{F})$.

then $\nu \ll \mu \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \mu(E) < \delta \Rightarrow |\nu(E)| < \varepsilon$.

Corollary If $f \in L^1(\mu)$, $\forall \varepsilon > 0, \exists \delta > 0$

$$\left| \int_E f \, d\mu \right| < \varepsilon \quad \text{whenever } \mu(E) < \delta.$$

Lemma. Suppose ν, μ finite measures on $(\Omega, \mathcal{F})$, Either

① $\nu \perp \mu$

② or $\exists \varepsilon > 0$, $E \in \mathcal{F}$ s.t. $\mu(E) > 0$ and $\nu \geq \varepsilon \mu$ on E.

Proof: Let $X = P_n \cup N_n$ be a Hahn decomposition for $\nu - \frac{1}{n}\mu$

Let $P = \bigcup_1^\infty P_n$ $N = \bigcap_1^\infty N_n = P^c$, Then. $\forall n \geq 1$.

$$0 \leq \nu(N) \leq \frac{1}{n}\mu(N)$$

It follows that $\nu(N) = 0$.

① If $\mu(P) = 0$, then $\nu \perp \mu$.

② If $\mu(P) > 0$, then $\exists N \in \mathbb{Z}_{>0}$, $\mu(P_N) > 0$, P_N is a positive set for $\nu - \frac{1}{n}\mu$.

Theorem. (Lebesgue - Radon - Nikodym theorem). Let ν σ-finite sign measure, μ σ-finite positive on $(\Omega, \mathcal{F})$. $\exists$ extended μ-integrable function $f: X \to \mathbb{R}$. and λ signed measure (σ-finite) on $(\Omega, \mathcal{F})$ such that

$$\underset{\text{a.e. unique.}}{f\,d\mu} \qquad d\nu = f\,d\mu + d\lambda, \quad \lambda \perp \mu.$$

$f\,d\mu$: a.e. unique. $d\lambda$: unique

Proof. Case I: Suppose ν, μ finite position measure. Let.

$$\mathcal{G} = \{ f : X \to [0,\infty] : \int_E f d\mu \leq \nu(E) \ \forall E \in \mathcal{F} \}.$$

Note that $\cdot$ $\mathcal{G} \neq \phi$ since $0 \in \mathcal{G}$

$\cdot$ $f, g \in \mathcal{G} \Rightarrow \max(f,g) \in \mathcal{G}$

Let $\alpha = \sup \left\{ \int f d\mu : f \in \mathcal{G} \right\} \leq \mu(\Omega) < \infty$.

choose $\{f_n\} \in \mathcal{F}$, $\int f_n d\mu \to \alpha$

Let $g_n = \max_{1 \leq i \leq n} \{f_i\}$, $f = \sup\{f_n\}$, then.

$$\int f_n d\mu \leq \int g_n d\mu \leq \alpha$$

$$\Rightarrow \int g_n d\mu \nearrow \alpha.$$

by monotone convergence theorem, $f \in \mathcal{F}$ and

$$\int f d\mu = \alpha.$$

Let

$$d\lambda = d\nu - f d\mu$$

Then we claim that $\lambda \perp \mu$.

If not, by Lemma, $\exists \ \varepsilon > 0, E \in \mathcal{F}, \lambda \geq \varepsilon\mu, \mu(E) > 0$.

But then.

$$\varepsilon \chi_E d\mu \leq d\lambda = d\nu - f d\mu.$$

$$\Rightarrow (f + \varepsilon \chi_E) d\mu \leq d\nu$$

$$\Rightarrow f + \varepsilon \chi_E \in \mathcal{F}$$

$$\Rightarrow \int (f + \varepsilon \chi_E) d\mu = \alpha + \varepsilon \mu(E) > \alpha$$

Contradicting the definition of α.

If $d\nu = d\lambda' + f' d\mu = d\lambda + f d\mu$, then.

$$d\lambda - d\lambda' = (f' - f)\, d\mu.$$

But. $d\lambda - d\lambda' \ll d\mu$, $(f'-f) d\mu \ll d\mu$, thus.

$$d\lambda - d\lambda' = (f' - f)\, d\mu = 0 \Rightarrow \lambda = \lambda', \quad f = f \text{ a.e.}$$

Case II: Suppose μ, ν are σ-finite measure. Then, $\exists \{A_j\} \subset \mathcal{F}$ s.t.

$$\mu(A_j), \nu(A_j) < \infty, \quad X = \bigsqcup_{j=1}^{\infty} A_j.$$

Define $\mu_j(E) = \mu(E \cap A_j)$, $\nu_j(E) = \nu(E \cap A_j)$

then $\exists \lambda_j, f_j$, $d\nu_j = d\lambda_j + f_j d\mu_j$

Since $\mu_j(A_j^c) = \nu_j(A_j^c) = 0$, we have

$$\lambda_j(A_j^c) = \nu_j(A_j^c) - \int_{A_j^c} f_j \, d\mu_j = 0.$$

and we can Assume $f_j = 0$ on A_j^c. Let $\lambda = \sum \lambda_j$, $f = \sum_i^{\infty} f_j$

Then $d\nu = d\lambda + f d\mu$, $\lambda \perp \mu$.

Case III: If ν is a signed measure. Apple to ν^+ and ν^-

Remark: $d\nu = \frac{d\nu}{d\mu} d\mu$ if $\nu \ll \mu$.

Proposition. Suppose that ν is a σ-finite signed measure and μ, λ are σ-finite measure on $(\Omega, \mathcal{F})$ such that $\nu \ll \mu$ and $\mu \ll \lambda$

①. If $g \in L^1(\nu)$, then $g\left(\frac{d\nu}{d\mu}\right) \in L^1(\mu)$ and.

$$\int g \, d\nu = \int g \frac{d\nu}{d\mu} d\mu.$$

②. $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} \quad \text{a.e.}$$

Proof ① By considering ν^+ and ν^-, we may assume that $\nu \geq 0$.

$$\mathcal{H} = \left\{ g \in L^1(\nu) : \int g \, d\nu = \int g \frac{d\nu}{d\mu} d\mu \right\}.$$

- $\mathcal{H}$: vector space.
- $\forall E \in \mathcal{F}$, $I_E \in \mathcal{H}$.
- $\{g_n\} \subset \mathcal{H}$, $g \in L^1$, $g_n \uparrow g \Rightarrow g \in \mathcal{H}$.

② $$\int g \, d\mu = \int g \frac{d\mu}{d\lambda} d\lambda.$$

Let $g = \chi_E \frac{d\nu}{d\mu}$. then.

$$\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int_E \frac{d\mu}{d\lambda} \frac{d\nu}{d\mu} d\lambda$$

But

$$\nu(E) = \int_E \frac{d\nu}{d\lambda} d\lambda.$$