黎曼几何报告

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2024 年 4 月 11 日

【Definition 1】 A C^∞ map $F : N \to M$ is said to be an *immersion* (a *submersion*) at $p \in N$ if its differential $\mathrm{d}F_p : T_pN \to T_{F(p)}M$ is injective (surjective).

【Example 2】 The inclusion of $\mathbb{R}^n$ in a higher dimensional $\mathbb{R}^m$

$$i\left(x^1, \cdots, x^n\right) = \left(x^1, \cdots, x^n, 0, \cdots, 0\right) \tag{1}$$

is an immersion.

The projection of $\mathbb{R}^n$ onto a lower-dimensional $\mathbb{R}^m$

$$\pi\left(x^1, \cdots, x^m, x^{m+1}, \cdots, x^n\right) = \left(x^1, \cdots, x^m\right) \tag{2}$$

is a submersion.

【Definition 3】 Let $F : N \to M$ be a smooth map of manifolds. Its *rank* at a point $p \in N$, denoted by $\mathrm{rank}F(p)$, is defined as the rank of the differential $dF_p : T_pN \to T_{F(p)}M$.

【Remark 4】 Relative to the coordinate neighbourhood $\left(U, x^1, \cdots, x^n\right)$ at p and $\left(V, y^1, \cdots, y^m\right)$ at $F(p)$, the differential is represented by the Jacobin matrix $[\partial F^i/\partial x^j(p)]$.

$$\text{i.e. } \mathrm{d}F_p\left(\left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^n}\right|_p\right) = \left(\left.\frac{\partial}{\partial y^1}\right|_p, \cdots, \left.\frac{\partial}{\partial y^m}\right|_p\right) J_F(p), \quad J_F(p) = \left(\frac{\partial F^j}{\partial x^i}(p)\right) = \left(\left.\frac{\partial y^i \circ F \circ \phi^{-1}}{\partial r^j}\right|_{\phi(p)}\right)$$

In summery, wehave

$$\mathrm{rank}F(p) = \dim\left(\mathrm{d}F_p(T_pN)\right) = \mathrm{rank}\left(\frac{\partial F^j}{\partial x^i}(p)\right)$$

【Theorem 5】(Constant rank theorem) Let N and M be manifolds of dimensions n and m respectively. Suppose $f : N \to M$ has constant rank k in a neighbourhood of a point p in N. Then there are charts (U, ϕ) near p in N and (V, ψ) near $f(p)$ in M such that

1. $\psi \circ f \circ \phi^{-1}\left(r^1, \cdots, r^n\right) = \left(r_1, \cdots, r^k, 0, \cdots, 0\right)$
2. $\phi(p) = \psi(f(p)) = 0$

【Proposition 6】 Let N and M be manifolds of dimensions n and m respectively. If a C^∞ map $f : N \to M$ is an immersion (submersion) at a point $p \in N$, then it has constant rank n (m) in a neighbourhood of p.

Proof. Let $(U, \phi) = \left(U, x^1, \cdots, x^n\right)$ be a chart about p in N and $(V, \psi) = \left(U, y^1, \cdots, y^n\right)$ be a chart about $f(p)$ in M. It follows that the map $\mathrm{d}f$ can be represented by the matrix $[\partial f^i/\partial x^j(p)]$, where

$$\frac{\partial f^i}{\partial x^j}(p) = \left.\frac{\partial}{\partial x_j}\right|_{\phi(p)} y^i \circ f \circ \phi$$

Hence

$$f \text{ is immersion at } p \iff \mathrm{d}f_p \text{ injective} \iff n \leq m \text{ and rank}\left(\frac{\partial f^i}{\partial x^j}(p)\right) = n$$

$$f \text{ is submersion at } p \iff \mathrm{d}f_p \text{ surjective} \iff n \geq m \text{ and rank}\left(\frac{\partial f^i}{\partial x^j}(p)\right) = m$$

It follows that

$$f \text{ is immersion or submersion at } p \iff f \text{ has maximal rank at } p.$$

We now assume that f has maximal rank k at p. Consider the subset of U:

$$D_{\max}(f) = \{p \in U : \mathrm{d}f \text{ has maximal rank at } p\}.$$

Since k is maximal, we have

$$\text{rank}(f)_p = k \iff \text{rank}\left(\frac{\partial f^i}{\partial x^j}(p)\right) = k \iff \text{rank}\left(\frac{\partial f^i}{\partial x^j}(p)\right) \geq kn$$

Thus

$$U \setminus D_{\max}(f) = \left\{p \in U : \text{rank}\left(\frac{\partial f^i}{\partial x^j}(p)\right) < n\right\}$$

which is equivalent to the vanishing of all $k \times k$ minors of the matrix $[\partial f^i/\partial x^j(p)]$. As the zero set of finitely many continuous functions, $U \setminus D_{\max}(f)$ is closed and thus $D_{\max}(f)$ is open. Then $D_{\max}$ is a neighbourhood of p which has constant rank k. And $k = n$ if f is an immersion, $k = m$ otherwise. □

Since an immersion or a submersion at p has a constant rank in a neighbourhood of p, by Constant rank theorem, we have:

【Theorem 7】 Let N and M be manifolds of dimensions n and m respectively.

1. **(Immersion theorem)** Suppose $f : N \to M$ is an immersion at p in N. Then there are charts (U, ϕ) near p and (V, ϕ) near $f(p)$ such that in a neighbourhood of $\phi(p)$,
   - $\phi \circ f \circ \phi^{-1}\left(r^1, \cdots r^n\right) = \left(r^1, \cdots r^n, 0 \cdots, 0\right)$
   - $\phi(p) = \psi(f(p)) = 0$
2. **(Submersion theorem)** Suppose $f : N \to M$ is an submersion at p in N. Then there are charts (U, ϕ) near p and (V, ϕ) near $f(p)$ such that in a neighbourhood of $\phi(p)$,
   - $\phi \circ f \circ \phi^{-1}\left(r^1, \cdots r^m, r^{m+1}, \cdots, r^n\right) = \left(r^1, \cdots r^m\right)$
   - $\phi(p) = \psi(f(p)) = 0$