

2.26.1

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$$f: A \rightarrow B \quad (A \text{ may be } \emptyset)$$

$$f(c) = \{f(x) : x \in c \cap A\}$$

$$f^{-1}(c) = \{x \in A : f(x) \in c\}.$$

$$f: A \rightarrow B \text{ injection.}$$

$$f: U \subset \mathbb{R}^n \rightarrow V \subset \mathbb{R}^m \quad (U \text{ open}).$$

$$f = (f^1, \dots, f^m)$$

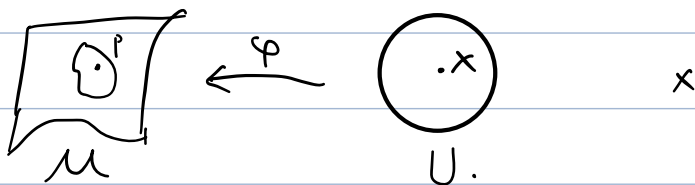
Def. If $f^i \in C^\infty(U)$, $i=1, \dots, m$, we say f is smooth.

U, V open. $f: U \rightarrow V$ bijective. $f, f^{-1} \in C^\infty$, f is a diffeomorphism.
($m=n$).

Local coordinate

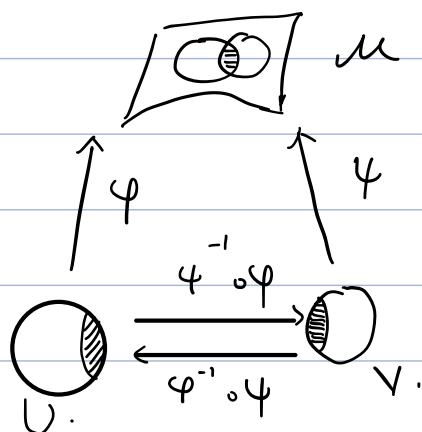
$$M \neq \emptyset. \quad \varphi: U \subset \mathbb{R}^n \rightarrow M \quad U: \text{open.} \quad \varphi \text{ injection.}$$

local coordinate on M .



$\text{Ran. } \varphi = M$: global coordinate.

$$\varphi: U \rightarrow M, \quad \psi: V \rightarrow M. \quad \text{local coordinate}$$



$\psi^{-1} \circ \varphi, \varphi^{-1} \circ \psi$: 坐标变换.

地图

If $\varphi^{-1} \circ \psi$ is a diffeomorphism, φ and ψ are compatible

$$\{\varphi_\lambda\}_{\lambda \in \Lambda}, \quad \mathcal{M} = \bigcup_{\lambda \in \Lambda} \text{Ran } \varphi_\lambda$$

Def 1: Let $\mathcal{M} \neq \emptyset$, $n \in \mathbb{Z}_{>0}$.

$$\varphi_\lambda: U_\lambda \rightarrow \mathcal{M}$$

is injective, $U_\lambda \subset \mathbb{R}^n$ is open, $\lambda \in \Lambda$.

Assume (i) $\mathcal{M} = \bigcup_{\lambda \in \Lambda} \text{R}(\varphi_\lambda)$

(ii) $\forall \lambda, \mu \in \Lambda$, $\varphi_\lambda^{-1} \circ \varphi_\mu: \text{Dom}(\varphi_\lambda^{-1} \circ \varphi_\mu) \rightarrow \text{R}(\varphi_\lambda^{-1} \circ \varphi_\mu)$
is a diffeomorphism.

Let $\mathcal{Q} = \{\varphi_\lambda: \lambda \in \Lambda\}$. We say $\mathcal{Q}$ is a diff structure on $\mathcal{M}$. $(\mathcal{M}, \mathcal{Q})$ is a n -d smooth manifold.

Example $(\mathbb{R}^n)$

$$\begin{aligned} I: \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto x \end{aligned}$$

$$\{I\}, \quad \mathbb{R}^n = (\mathbb{R}^n, \{I\})$$

Topology on manifold.

Let $\Omega \subset \mathcal{M}$. If $\varphi^{-1}(\Omega)$ is open ($\forall \varphi \in \mathcal{Q}$), we say Ω is open.

$\mathcal{T} = \{\Omega \subset \mathcal{M}: \varphi^{-1}(\Omega) \text{ open for all } \varphi \in \mathcal{Q}\}$ is a topology on $\mathcal{M}$.

(A). Hausdorff axiom. $\forall x, y \in \Omega$, $x \neq y$, $\exists U \in \mathcal{N}_x$, $V \in \mathcal{N}_y$, $U \cap V = \emptyset$.

(B). Countable basis axiom.

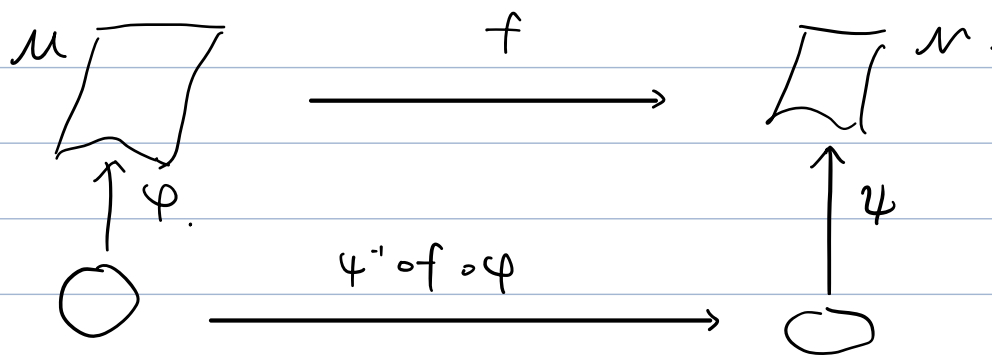
$$\exists \varphi_k \in \mathcal{Q}, k \in \mathbb{Z}_{>0}, \quad \mathcal{M} = \bigcup_{k=1}^{\infty} \text{R}(\varphi_k).$$

Smooth maps.

$\{\mathcal{M}, \mathcal{Q}_\mathcal{M}\}$, $\{\mathcal{N}, \mathcal{Q}_\mathcal{N}\}$ manifolds. $f: \mathcal{M} \rightarrow \mathcal{N}$ continuous.

Let $\varphi \in \mathcal{Q}_\mathcal{M}$, $\psi \in \mathcal{Q}_\mathcal{N}$

$\psi^{-1} \circ f \circ \varphi$ coordinate expression of f .



If $\forall \varphi \in \mathcal{D}_M, \psi \in \mathcal{D}_N, \psi^{-1} \circ f \circ \varphi$ is smooth, we say f is smooth.

- $C^\infty(M, N)$: set of smooth maps from M to N .
- $C^\infty(M, \mathbb{R}) = C^\infty(M)$

Def. $f \in C^\infty(M, N)$. If

- ①. f is bijective
- ②. $f, f^{-1} \in C^\infty$

We say that f is a diffeomorphism. $\dim M = \dim N$.

If $\exists$ a diffeomorphism $f: M \rightarrow N$, we say M and N are diffeomorphic.
 $M \sim N$

M and N are diffeomorphic. $\Rightarrow M$ and N homeomorphic.
 $\Leftarrow$ (x).

Milnor's 7-sphere

Maximal differentiable structure.

$(M, \mathcal{D})$

$\overline{\mathcal{D}} = \{ \varphi: U \rightarrow M : U \subset \mathbb{R}^n \text{ open, } \varphi \text{ injective, } \forall \psi \in \mathcal{D}, \psi^{-1} \circ \varphi \text{ is a diffeomorphism} \}$.

$(\supset \mathcal{D})$

$\overline{\mathcal{D}}$ is differentiable structure.

$$(\mathcal{M}, \mathcal{D}) = (\mathcal{M}, \overline{\mathcal{D}}) \quad (\text{diffeomorphism}).$$

We say $\mathcal{D}$ is maximal differentiable structure on $\mathcal{M}$.

If $\varphi \in \overline{\mathcal{D}}$, φ : local coordinate on $\mathcal{M}$.

$\mathcal{M}$ manifold $p \in \mathcal{M}$. $\exists \varphi: B \subset \mathbb{R}^n \rightarrow \mathcal{M}$, $\varphi(p) = 0$.

$\exists \varphi: \mathbb{R}^n \rightarrow \mathcal{M}$, $\varphi(p) = 0$.

Open submanifolds

$(\mathcal{M}, \mathcal{D})$, $\Omega \subset \mathcal{M}$ open, nonempty

$\mathcal{D}_\Omega = \{ \varphi|_{\varphi^{-1}(\Omega)} : \varphi \in \mathcal{D} \}$ is a differentiable structure on Ω

$(\Omega, \mathcal{D}_\Omega)$ open submanifolds of $(\mathcal{M}, \mathcal{D})$

$f: \mathcal{M} \rightarrow \mathcal{N}$.

Let $\Omega \subset \mathcal{M}$ be open.

If $f: \Omega \rightarrow \mathcal{N}$ is smooth, we say f is smooth in Ω .

Let $p \in \mathcal{M}$. If $\exists U \in \mathcal{N}(p)$, s.t. f smooth in U , we say f is smooth at p .

$\varphi \in \mathcal{D}$, $\text{Dom } \varphi = U$, $R(\varphi) = V$, $\varphi: U \rightarrow V$ bijection.

φ is a diffeomorphism.

$f: \mathcal{M} \rightarrow \mathbb{R}$

$C_0^\infty(\mathcal{M}) = \{ f \in C^\infty(\mathcal{M}) : \text{supp}(f) \text{ compact} \}$.

Theorem. $\mathcal{M}$ is a connected manifold. $\mathcal{O}$ is a set of open subsets of $\mathcal{M}$,

$$\mathcal{M} = \bigcup_{V \in \mathcal{O}} V$$

Then, $\exists \varphi_\lambda \in C_0^\infty(\mathcal{M})$, $0 \leq \varphi \leq 1$, $\lambda \in \Lambda$ s.t.

- ① $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is local finite, $\forall x_0 \in M, \exists V \in \mathcal{N}(x_0), \lambda_i \in \Lambda, i=1, \dots, N$
such that $\text{supp } \varphi_\lambda \cap V = \emptyset, \forall \lambda \neq \lambda_i, i=1, \dots, N$
- ② $\forall \lambda \in \Lambda, \exists V \in \mathcal{O}$, such that $\text{supp } \varphi_\lambda \subset V$
- ③. $\sum_{\lambda \in \Lambda} \varphi_\lambda(x) = 1 \quad \forall x \in M.$

Corollary. Let $p \in M, V \in \mathcal{N}(p)$. then $\exists \varphi \in C_c^\infty(M), 0 \leq \varphi \leq 1$ such that

①. $\text{supp } \varphi \subset V$

cut-off function.

② $\varphi \equiv 1$ in some neighbourhood V of p .

Example (Linear space.)

X : linear space $\dim X = n$

$$X = \text{span} \{v_i\}_{i=1}^n$$

define

$$\varphi: \mathbb{R}^n \rightarrow X$$

$$x \mapsto \sum_{i=1}^n x^i v_i \quad x = (x^1, \dots, x^n)$$

$$= x^i v_i$$

$$\mathcal{D} = \{\varphi\} \text{ differentiable structure } X = (X, \mathcal{D})$$

Example (Graph of smooth function)

$$f: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m.$$

Ω open.

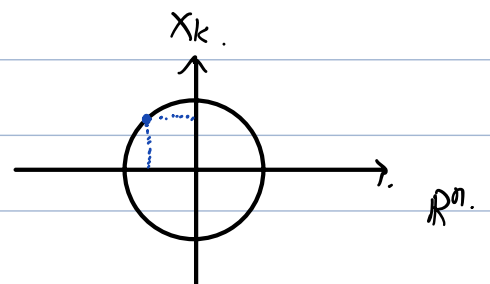
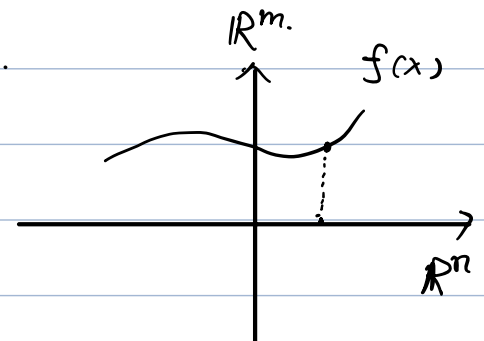
$$G(f) = \{(x, f(x)) \in \mathbb{R}^{n+m} : x \in \Omega\}$$

define

$$\varphi: \Omega \rightarrow G(f)$$

$$x \mapsto (x, f(x)).$$

$$\mathcal{D} = \{\varphi\}, \quad G(f) \text{ is a } n \text{ d manifold}$$



Example (n -sphere)

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}.$$

for all k ($1 \leq k \leq n+1$)

$$\varphi_k(x): B_1 \longrightarrow S^n \quad B_1 = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

$$x \longmapsto (x^1, \dots, x^{k-1}, \sqrt{1-|x|^2}, x^k, \dots, x^{n+1})^T$$

$$\psi_k(x): B_1 \longrightarrow S^n$$

$$x \longmapsto (x^1, \dots, x^{k-1}, -\sqrt{1-|x|^2}, x^k, \dots, x^{n+1})^T$$

$$\mathcal{D} = \{\varphi_k\} \cup \{\psi_k\}.$$

Example

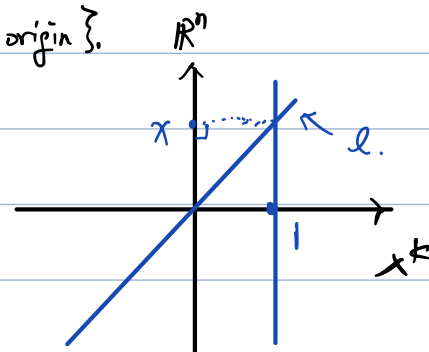
$\mathcal{P}_n = \{l : l \text{ is a straight line in } \mathbb{R}^{n+1} \text{ passing through origin}\}.$

$$k = 1, 2, \dots, n+1$$

$$\varphi_k(x) = \{tX : t \in \mathbb{R}\}$$

$$\text{where } X = (x^1, \dots, x^{k-1}, 1, x^{k+1}, \dots, x^{n+1})^T$$

$$\mathcal{D} = \{\varphi_k\}_{k=1}^{n+1}$$



Example (Product manifold)

$$(\mathcal{M}, \mathcal{D}_\mathcal{M}) \quad (\mathcal{N}, \mathcal{D}_\mathcal{N}) \quad \dim \mathcal{M} = m \quad \dim \mathcal{N} = n$$

$$\varphi \in \mathcal{D}_\mathcal{M}, \quad \psi \in \mathcal{D}_\mathcal{N}.$$

$$\text{define: } \varphi \times \psi : D(\varphi) \times D(\psi) \longrightarrow \mathcal{M} \times \mathcal{N}.$$

$$(x, y) \longmapsto (\varphi(x), \psi(y))$$

$$\text{then } \mathcal{D}_{\mathcal{M} \times \mathcal{N}} = \{\varphi \times \psi : \varphi \in \mathcal{D}_\mathcal{M}, \psi \in \mathcal{D}_\mathcal{N}\}$$

Differential of smooth map/ function.

$$p \in \mathcal{M}.$$

$$C_p^\infty = \{f \in C^\infty(U) : U \in \mathcal{N}_p\} \rightsquigarrow \text{algebra}.$$

$$\text{If } f, g \in C_p^\infty, \quad f \equiv g \text{ in some neighbourhood of } p \quad f = g$$

$$\gamma \in C^\infty((a,b), \mathcal{M}) \quad \text{smooth curve on } \mathcal{M}.$$

Definition Let $\gamma \in C^\infty((a,b), M)$, $\gamma(t_0) = p$. Define.

$$\langle \gamma'(t_0), f \rangle = \left. \frac{d}{dt} \right|_{t=t_0} f \circ \gamma \quad \forall f \in C_p^\infty.$$

We call $\gamma'(t_0)$ the tangent of γ at t_0 .

$$\textcircled{1} \text{ If } \exists U \in \mathcal{N}(p), \quad f = g \text{ in } U. \quad \left. \frac{d}{dt} \right|_{t=t_0} f \circ \gamma = \left. \frac{d}{dt} \right|_{t=t_0} g \circ \gamma$$

$$\textcircled{2} \quad \langle \gamma'(t_0), \alpha f + \beta g \rangle = \alpha \langle \gamma'(t_0), f \rangle + \beta \langle \gamma'(t_0), g \rangle.$$

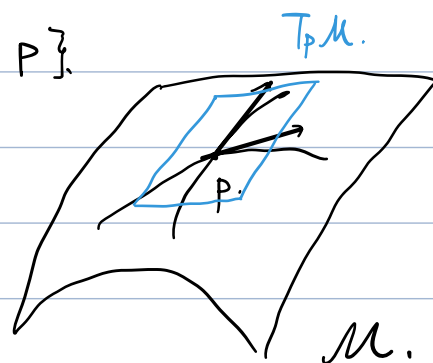
$$\Rightarrow \gamma'(t_0) \in \mathcal{L}(C_p^\infty, \mathbb{R}).$$

$$\text{Def } T_p M = \{ \gamma'(t_0) : \gamma \in C^\infty((a,b); M), \gamma(t_0) = p \}.$$

tangent space of M at p .

$v \in T_p M$: tangent vector of M at p .

$$\text{If } p \neq q, \quad T_p M \cap T_q M = \emptyset$$



Theorem $T_p M$ is a n -d linear subspace of $\mathcal{L}(C_p^\infty, \mathbb{R})$.

Let φ be a local coordinate around p $\varphi = \varphi(x)$

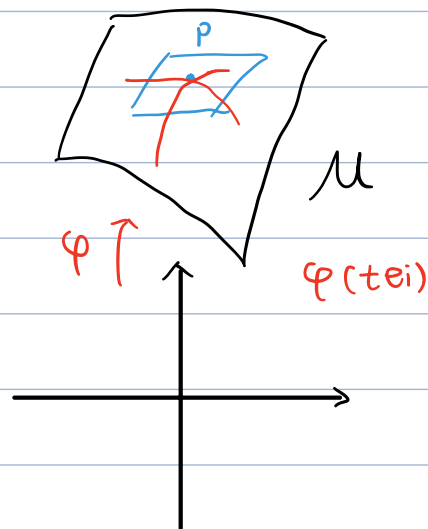
$$p = \varphi(x_0)$$

$$\left. \frac{\partial}{\partial x^i} \right|_p = \left. \frac{d}{dt} \right|_{t=t_0} \varphi(x_0 + t e_i)$$

$$\langle \left. \frac{d}{dt} \right|_{t=t_0} \varphi(x_0 + t e_i), f \rangle$$

$$= \left. \frac{d}{dt} \right|_{t=t_0} f(\varphi(x_0 + t e_i))$$

$$= \frac{\partial f \circ \varphi}{\partial x^i}(x_0)$$



$$\text{Theorem: } T_p M = \text{span} \left\{ \left. \frac{\partial}{\partial x^i} \right|_p \right\}_{i=1}^n$$

Proof. Let $\gamma: (a,b) \rightarrow M$ $\gamma(t_0) = p$.

$$\gamma^i = (\varphi^{-1} \circ \gamma)^i$$

Claim:

$$r'(t_0) = (r^i(t_0))' \frac{\partial}{\partial x^i} \Big|_p$$

$$\forall f \in C_p^\infty$$

$$\langle r'(t_0), f \rangle = \frac{d}{dt} \Big|_{t=t_0} f \circ r = \frac{d}{dt} \Big|_{t=t_0} (f \circ \varphi \circ \varphi^{-1} \circ r)$$

$$\begin{aligned} \langle (r^i(t_0))' \frac{\partial}{\partial x^i} \Big|_p, f \rangle &= (r^i(t_0))' \frac{\partial f}{\partial x^i} \Big|_p \\ &= (r^i(t_0))' \frac{\partial f \circ \varphi}{\partial x^i} (x_0). \end{aligned}$$

3.11

Let $v \in T_p M$, $f \in C_p^\infty$

Prop. Let $f, g \in C_p^\infty$, $v, w \in T_p M$. $\alpha, \beta \in \mathbb{R}$.

①. If $f = g$ in some neighbourhood of p , then,

$$\langle v, f \rangle = \langle v, g \rangle.$$

$$\textcircled{2} \quad \langle v, \alpha f + \beta g \rangle = \alpha \langle v, f \rangle + \beta \langle v, g \rangle.$$

$$\textcircled{3} \quad \langle \alpha v + \beta w, f \rangle = \alpha \langle v, f \rangle + \beta \langle w, f \rangle$$

$$\textcircled{4} \quad \langle v, fg \rangle = f(p) \langle v, g \rangle + g(p) \langle v, f \rangle.$$

THM: X ind vector space, $\Omega \subset X$ open, $p \in \Omega$, then

$$T_p \Omega \cong X.$$

Proof. define

$$L: X \longrightarrow T_p \Omega$$

$$v \longmapsto \frac{d}{dt} \Big|_0 (p + tv).$$

$$\text{认为 } v = \frac{d}{dt} \Big|_0 (p + tv) \quad (\text{同构意义})$$

$$X = T_p \Omega$$

(Claim. L is an isomorphism.

$$\text{Let } X = \text{span} \{E_i\}_{i=1}^n$$

$$\varphi(x) = \sum x^i E_i$$

$$\left\{ \frac{\partial}{\partial x^i} \Big|_p \right\} \text{ basis of } T_p \Omega.$$

It suffices to check that.

$$L(v^i E_i) = v^i \frac{\partial}{\partial x^i} \Big|_p$$

In fact:

$$L(\sum v^i E_i) = \frac{d}{dt} \Big|_0 (p + t \sum v^i E_i)$$

$$\begin{aligned}
&= \frac{d}{dt}\bigg|_0 \sum (p_i + tv_i) E_i \\
&= \sum \alpha_i \frac{\partial}{\partial x_i}\bigg|_p \\
\Rightarrow \quad \alpha_i &= \left\langle \frac{d}{dt}\bigg|_0 \sum (p_i + tv_i) E_i, X^i \right\rangle \\
&= \frac{d}{dt}\bigg|_0 (p_i + tv_i) = v_i \\
\text{i.e.} \quad L(\sum v_i E_i) &= \sum v_i \frac{\partial}{\partial x_i}\bigg|_p
\end{aligned}$$

Differential.

Definition: $f: M \rightarrow N$ smooth, $p \in M$, differential of f at p is a map.

$$df_p: T_p M \rightarrow T_{f(p)} N$$

$$df_p(v) = \frac{d}{dt}\bigg|_0 f \circ \gamma.$$

where.

$$\gamma(0) = p \quad \gamma'(0) = v.$$

$$\forall X \in T_p M.$$

$$df_p: T_p M \rightarrow T_p N$$

$$\begin{aligned}
df_p(v) &= \frac{d}{dt}\bigg|_0 f \circ \gamma \\
&= \sum \alpha_i \frac{\partial}{\partial y_i}
\end{aligned}$$

$$\text{where } \alpha_i = \left\langle \frac{d}{dt}\bigg|_0 f \circ \gamma, Y_i \right\rangle.$$

$$\begin{aligned}
&= \left\langle \frac{d}{dt}\bigg|_0, Y_i \circ f_p \circ \gamma \right\rangle \\
&= \left\langle \frac{d}{dt}\bigg|_0, f^i(\gamma_1(t), \dots, \gamma_n(t)) \right\rangle \\
&= \sum \frac{\partial f^i}{\partial x_j} \circ \gamma_j(0) \\
&= \sum \frac{\partial f^i}{\partial x_j} v_j
\end{aligned}$$

$$\text{Chain rule. } d(g \circ f)_p = (dg)_{f(p)} \circ (df)_p.$$

$$\forall X \in T_p M. \quad c(0) = p, c'(0) = X$$

$$\begin{aligned}
 d(g \circ f)_p(X) &= \left. \frac{d}{dt} \right|_0 g \circ f \circ c \\
 &= dg \left(\left. \frac{d}{dt} \right|_0 f \circ c \right) \\
 &= dg \circ df \left(\left. \frac{d}{dt} \right|_0 c \right) \\
 &= dg \circ df(X).
 \end{aligned}$$

1. Let $I_M : M \rightarrow M$ be identity. Prove $d(I_M)_p = I_{T_p M}$

2. Let $f \in C^\infty(M)$, Then,

$$df_p(v) = \langle v, f \rangle \quad \forall v$$

3. Let $\gamma : (a, b) \rightarrow M$ smooth, then

$$d\gamma_{t_0}(1) = \gamma'(t_0)$$

Proof. 1. $\forall X \in T_p M$, let $c(t) = p$ $c'(t) = X$.

$$\begin{aligned}
 (dI_M)_p(X) &= \left. \frac{d}{dt} \right|_0 I_M \circ c \\
 &= \left. \frac{d}{dt} \right|_0 c = c'(t) = X
 \end{aligned}$$

$$\text{i.e. } (dI_M)_p = I_{T_p M}.$$

$$\begin{aligned}
 2. \quad df_p(v) &= \left. \frac{d}{dt} \right|_0 f \circ c = \left\langle \left. \frac{d}{dt} \right|_0 c, f \right\rangle \\
 &= \langle v, f \rangle
 \end{aligned}$$

$$3. \quad (d\gamma)_{t_0} \left(\left. \frac{d}{dt} \right|_0 \right) = \left. \frac{d}{dt} \right|_0 \gamma$$

Tangent bundle.

$$TM = \bigcup_{p \in M} T_p M.$$

Let φ local coordinate on M , $\text{Dom } \varphi = U \subset \mathbb{R}^n$, $R(\varphi) = \Omega$

$$\hat{\varphi}(x, v) = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_{\varphi(x)} \quad x, v \in \mathbb{R}^n.$$

where.

$$\frac{\partial}{\partial x^i} \Big|_{\varphi(p)} = \frac{d}{dt} \Big|_0 \varphi(\varphi^{-1}(p) + te_i)$$

$$\hat{\mathcal{D}} = \{ \hat{\varphi} : \varphi \in \mathcal{D} \}. \quad \text{differential struct}$$

$$(T\mathcal{M}, \hat{\mathcal{D}}) \rightsquigarrow \text{tangent bundle.}$$

Def: Let $X: \mathcal{M} \rightarrow T\mathcal{M}$ be smooth if

$$X(p) \in T_p \mathcal{M} \quad \text{for all } p \in \mathcal{M}$$

we say X is a smooth vector field.

$$\text{Def: } \frac{\partial}{\partial x^i} : \Omega \rightarrow T_p \Omega.$$

$$p \mapsto \frac{\partial}{\partial x^i} \Big|_p = \frac{d}{dt} \Big|_0 \varphi(\varphi^{-1}(p) + te_i)$$

$\frac{\partial}{\partial x^i}$ is a smooth vector field on Ω . $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ basis vector field

$$(\hat{\varphi}^{-1} \frac{\partial}{\partial x^i} \circ \varphi)(x) = (x, e_i) \in \mathbb{C}^\infty$$

X smooth vector field

$$X = \sum X^i \frac{\partial}{\partial x^i} \quad \text{in } \Omega.$$

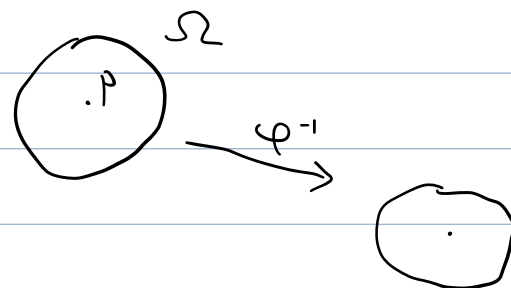
$X^i : \Omega \rightarrow \mathbb{R}$ components of X , $X^i \in C^0(\Omega)$.

$$(\hat{\varphi}^{-1} \circ X \circ \varphi)(x) = (x, X^1 \circ \varphi, \dots, X^n \circ \varphi) \quad \text{smooth}$$

$$\Rightarrow X^i \circ \varphi \text{ smooth.}$$

$$\Rightarrow X^i \text{ smooth.}$$

$$\text{Prop } T\Omega = \Omega \times \mathbb{R}^n.$$



$$\psi : T\Omega \rightarrow \Omega \times \mathbb{R}^n.$$

$$\sum v^i \frac{\partial}{\partial x^i} \Big|_p \mapsto (p, \sum v^i e_i)$$

$$T\mathbb{S}^n.$$

$$\mathcal{V}(M) = \{X : X \text{ smooth vector field on } M\}.$$

$$\text{Let } f, g \in C^\infty(M), X, Y \in \mathcal{V}(M).$$

$$(fX + gY)_p = f(p)X_p + g(p)Y_p \quad p \in M$$

$$\Rightarrow fX + gY \in \mathcal{V}(M).$$

$$\mathcal{V}(M) : C^\infty(M) \text{ - module.}$$

Action of vector field on smooth function.

$$X \in \mathcal{V}(M), f \in C^\infty(M). \text{ Define}$$

$$(Xf)(p) = \langle X_p, f \rangle. \quad \text{for all } p \in M.$$

$$Xf \in C^\infty(M).$$

Prop. Let $X, Y \in \mathcal{V}(M), f, g, h \in C^\infty(M), \alpha, \beta \in \mathbb{R}$

$$\textcircled{1} (fX + gY)(h) = fXh + gYh.$$

$$\textcircled{2} X(\alpha f + \beta g) = \alpha Xf + \beta Xg.$$

$$\textcircled{3} X(fg) = fXg + gXf$$

X : linear differential operator of 1 - order



vector field.

$$\text{Let } X, Y \in \mathcal{V}(M)$$

$$XY : C^\infty(M) \longrightarrow C^\infty(M).$$

$$f \longmapsto X(Yf)$$

linear differential operator of 2 - order.

$$X, Y \in \mathcal{V}(M)$$

$$XY - YX \quad \text{commutator.}$$

1-order

Prop Let $X, Y \in \mathcal{V}(M)$ Then. $\exists! [X, Y] \in \mathcal{V}(M)$ such that
$$[X, Y]f = (XY - YX)f \quad \text{for all } f \in C^\infty(M).$$

$[X, Y]$: Lie bracket of X, Y .

Prop: Let $f \in C^\infty(M)$
$$\frac{\partial f}{\partial x^i} \circ \varphi = \frac{\partial (f \circ \varphi)}{\partial x^i}$$

Proof. $\frac{\partial}{\partial x^i} \Big|_p f = \frac{d}{dt} f \circ \varphi(\varphi^{-1}(p) + e^{it})$
$$= \frac{\partial (f \circ \varphi)}{\partial x^i} \Big|_{\varphi^{-1}(p)} = \frac{\partial (f \circ \varphi)}{\partial x^i} \circ \varphi^{-1}(p)$$

Prop Let $X = X^i \frac{\partial}{\partial x^i}$, $f \in C^\infty(M)$
$$Xf = X^i \frac{\partial f}{\partial x^i}$$

Prop $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$.

Proof $(\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i}) f \circ \varphi = \frac{\partial^2 f \circ \varphi}{\partial x^i \partial x^j} - \frac{\partial^2 f \circ \varphi}{\partial x^j \partial x^i} = 0$.

Prop. Let. $X = \sum x^i \frac{\partial}{\partial x^i}$ $Y = \sum y^j \frac{\partial}{\partial x^j}$
then

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^j}$$

Proof.
$$\left(\sum x^i \frac{\partial}{\partial x^i} \right) \left(\sum y^j \frac{\partial}{\partial x^j} \right) - \left(\sum y^j \frac{\partial}{\partial x^j} \right) \left(\sum x^i \frac{\partial}{\partial x^i} \right)$$

$$= \sum_i \sum_j x^i \frac{\partial}{\partial x^i} \left(y^j \frac{\partial}{\partial x^j} \right) - \sum_j \sum_i y^j \frac{\partial}{\partial x^j} \left(x^i \frac{\partial}{\partial x^i} \right)$$

$$= \sum_i \sum_j \left(x^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial x^j} + x^i y^j \frac{\partial^2}{\partial x^i \partial x^j} \right) - \sum_j \sum_i \left(y^j \frac{\partial x^i}{\partial x^j} \frac{\partial}{\partial x^i} + x^i y^j \frac{\partial^2}{\partial x^j \partial x^i} \right)$$

$$= \sum_i \sum_j x^i \frac{\partial x^j}{\partial x^i} \frac{\partial}{\partial x^j} - \sum_j \sum_i y^j \frac{\partial x^i}{\partial x^j} \frac{\partial}{\partial x^i}$$

Prop: $X, Y, Z \in \mathcal{V}(M)$

①. $[X, Y] = -[Y, X]$.

②. $[\alpha X + \beta Y, Z] = \alpha [X, Z] + \beta [Y, Z]$.

③. $[[X, Y], Z] + [[Z, X], Y] + [[Z, Y], X] = 0$ Jacobi
非交换算子对称性.

$(\mathcal{V}(M), [\cdot, \cdot])$ Lie algebra.

Proof. ③. $[[X, Y], Z] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX$

$F: M \rightarrow N$ is a diffeomorphism.

Def. Let $X \in \mathcal{V}(M)$,

$$dF(X)(F(p)) = dF_p(X(p)). \quad p \in M.$$

Prop. Let $F: M \rightarrow N$ be a diffeomorphism. Then.

$$dF([X, Y]) = [dF(X), dF(Y)]$$

Proof. $\forall g \in C^\infty(M)$.

$$\begin{aligned} \langle dF([X, Y]), g \rangle &= \langle [X, Y], g \circ F \rangle \\ &= \left\langle \left(X^i \frac{\partial x^j}{\partial x^i} - Y^i \frac{\partial x^j}{\partial x^i} \right) \frac{\partial}{\partial x^i}, g \circ F \right\rangle \\ &= \left\langle \left(X^i \frac{\partial x^j}{\partial x^i} - Y^i \frac{\partial x^j}{\partial x^i} \right) dF \frac{\partial}{\partial x^i}, g \right\rangle \\ &= \langle [dF(X), dF(Y)], g \rangle \end{aligned}$$

Def. Let $B: \mathcal{V}(M) \times \mathcal{V}(M) \rightarrow C^\infty(M)$. If.

$$\bullet B(fX + gY, Z) = f B(X, Z) + g B(Y, Z).$$

$$\bullet B(X, fY + gZ) = f B(X, Y) + g B(X, Z).$$

We say B is a bilinear form.

Prop Let $F: \mathcal{V}(M) \rightarrow C^\infty(M)$ such that

$$F(fX + gY) = fF(X) + gF(Y) \quad \forall f, g \in C^\infty(M)$$

Let $p \in M$, $X, \tilde{X} \in \mathcal{V}(M)$, If

$$X(p) = \tilde{X}(p).$$

function.



F - linear.

then. $F(X)(p) = F(\tilde{X})(p)$.

~~Proof~~. Let $X(p) = 0$. $X = \sum X^i \frac{\partial}{\partial x^i}$ on $U \in \mathcal{N}(p)$.

Let $f(p) = 1$, $\text{supp } f \subset U$.

then.

$$\begin{aligned} f^2 F(X) &= F(f^2 X) = F(f^2 \sum X^i \frac{\partial}{\partial x^i}) \\ &= \sum f X^i F(f \frac{\partial}{\partial x^i}) \end{aligned}$$

thus. $X(p) = 0 \Rightarrow X^i(p) = 0$.

$$\Rightarrow F(X)(p) = 0.$$

Claim Let $B: \mathcal{V}(M) \times \mathcal{V}(M) \rightarrow C^\infty(M)$ be a bilinear form, $p \in M$.

If $X(p) = \tilde{X}(p)$, $Y(p) = \tilde{Y}(p)$ then.

$$B(X, Y)(p) = B(\tilde{X}, \tilde{Y})(p).$$

Let B be a bilinear form on M . $p \in M$, $u, v \in T_p M$.

$$B(u, v) \stackrel{\Delta}{=} B(X, Y)(p) \quad X, Y \in \mathcal{V}(M).$$

$$X(p) = u \quad Y(p) = v.$$

$B: T_p M \times T_p M \rightarrow \mathbb{R}$ bilinear form

Let $\Omega \subset M$ be open, $X, Y \in \mathcal{V}(\Omega)$

$$B(X, Y)(p) = B(X|_p, Y|_p) \quad \forall p \in \Omega.$$

$$B \in C^\infty(\Omega)$$

$$B: \mathcal{V}(\Omega) \times \mathcal{V}(\Omega) \rightarrow C^\infty(\Omega) \quad \text{bilinear form.}$$

Ω is a coordinate neighborhood.

$$\frac{\partial}{\partial x^i} \text{ basis.} \quad B\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) =: B_{ij} \quad \leadsto \text{component of } B$$

$$X = \sum X^i \frac{\partial}{\partial x^i}, \quad Y = \sum Y^j \frac{\partial}{\partial x^j} \quad \text{in } \Omega.$$

$$B(X, Y) = \sum_{i,j} X^i Y^j B\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_{i,j} X^i Y^j B_{ij}$$

g : bilinear form on $\mathcal{M}$. If

$$g(u, v) = g(v, u) \quad \text{for all } u, v \in T_p M, \quad p \in M.$$

We say: g is symmetric. If.

$$g(u, u) > 0 \quad \forall u \in T_p M, u \neq 0, \quad p \in M.$$

We say: g is positive.

Def. Let g be a symmetric, positive bilinear form on $\mathcal{M}$. We say g is a Riemannian metric on M .

(M, g) Riemann manifold.

欧氏空间上的 Riemann metric: $g(x, y) = x^T y$.

Let $u, v \in T_p M$

inner product on $T_p M$

• $g(u, v)$: inner product

$$g: T_p M \times T_p M \rightarrow \mathbb{R}.$$

• $\sqrt{g(u, u)} = |u|$: norm

• g_{ij} : 度规系数 $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

(g_{ij}) : 度规矩阵. 对称, 正定.

g : Riemann metric on M .

Let $\gamma : [a, b] \rightarrow M$ be continuous.

• If $\exists \tilde{\gamma} : (a', b') \rightarrow M$, $\tilde{\gamma}$ is smooth

$$\tilde{\gamma}|_{[a, b]} = \gamma.$$

We say γ is a smooth segment.

• If $\exists$ a partition

$$a = a_1 < a_2 < \dots < a_N = b$$

such that $\gamma|_{[a_i, a_{i+1}]}$ is smooth ($1 \leq i \leq N-1$), we say γ is piecewise smooth segment.

Def. Let $\gamma : [a, b] \rightarrow M$ be a piecewise smooth segment

$$L(\gamma) = L(\gamma; [a, b]) = \int_a^b |\gamma'(t)| dt$$

Let $p, q \in M$. Define.

Assume M is connected

$$d(p, q) = \inf L(\gamma)$$

γ is piecewise smooth with $\gamma(a) = p$, $\gamma(b) = q$.

Claim d is a metric on M .

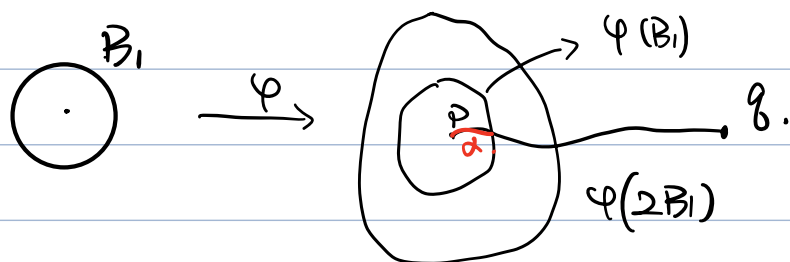
Let $p, q \in M$, $p \neq q$, let $\varphi : B_2 \rightarrow M$ be a local coordinate

$$\varphi^{-1}(\varphi) = 0, \quad q \notin R(\varphi).$$

let $\gamma : [a, b] \rightarrow M$. $\gamma(a) = p$, $\gamma(b) = q$.

$\exists c \in [a, b]$, $\gamma(c) \in \varphi(B_1)$

$$\gamma(t) \in \varphi(B_1) \quad t \in [a, c].$$



let $\alpha = \gamma|_{[a, c]}$. $L(\gamma) \geq L(\alpha)$ let $\beta = \varphi^{-1}\alpha$.

$$\beta(a)=0 \quad \beta(c) \in \partial B_1$$

$$L(\alpha) = \int_a^c |\alpha'(t)| dt.$$

$$\alpha'(t) = \sum (\beta^i)'(t) \frac{\partial x^i}{\partial x^i} |_{\alpha(t)}.$$

$$|\alpha'(t)|^2 = \sum_{i,j} (\beta^i)'(t) (\beta^j)'(t) g_{ij}(\alpha(t))$$

令 λ_1 为 (g_{ij}) 的最小特征值

$$\lambda_1 : R(\varphi) \rightarrow R.$$

$\lambda_1 > 0$. is continuous.

$$\Rightarrow |\alpha'(t)|^2 \geq \lambda_1(\alpha(t)) |\beta'(t)|^2 \geq c_0^2.$$

(M, d) metric space.

完备 $\Leftrightarrow$ 测地线无限长. $\mathcal{H}_{opt}$ - Riemann.

定义积分.

Setup 1 Integral coordinate neighbourhood.

Ω coordinate neighbourhood. $\varphi: \varphi^{-1}(\Omega) \rightarrow \Omega$

$p \in \Omega$. $\left\{ \frac{\partial x^i}{\partial x^i} |_p \right\}$. basis of $T_p M$.

$$Q = \left\{ \sum t^i \frac{\partial x^i}{\partial x^i}(\varphi) : 0 \leq t^i \leq 1 \right\}.$$

$$V(Q) = \frac{1}{n!} \left| \det \left(g \left(\frac{\partial x^i}{\partial x^i}(\varphi), \frac{\partial x^j}{\partial x^j}(\varphi) \right) \right) \right|$$

$$C_0(\Omega) = \{ f \in C(\Omega) : \text{supp}(f) \text{ compact} \}.$$

$$I_\Omega(f) = \int_{\varphi^{-1}(\Omega)} f \circ \varphi \sqrt{\det g_{ij} \circ \varphi} \quad f \in C_0(\Omega)$$

volume element.

y : coordinate on Ω .

$$\frac{\partial}{\partial y^i} = d\gamma\left(\frac{\partial}{\partial x^i}\right)$$

$$\begin{aligned} g_{ij}(p) &= g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\bigg|_p \\ &= \sum_{k,l} \hat{g}_{kl} dy^k \otimes dy^l \left(\sum \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial y^k}, \sum \frac{\partial y^l}{\partial x^j} \frac{\partial}{\partial y^l} \right) \\ &= \sum_{k,l} \hat{g}_{kl} \frac{\partial y^k}{\partial x^i}(p) \frac{\partial y^l}{\partial x^j}(p). \end{aligned}$$

$$\Rightarrow \sqrt{\det(g_{ij}(p))} = \left| \det \frac{\partial y^j}{\partial x^i}(p) \right| \cdot \sqrt{\det \hat{g}_{ij}(p)}$$

Prop 1

$$(i) \quad I_{\Omega}(f) \geq 0 \quad \forall f \in C_0(\Omega)$$

$$(ii) \quad I_{\Omega}(\alpha f + \beta g) = \alpha I_{\Omega}(f) + \beta I_{\Omega}(g)$$

Step. Integral of continuous function with compact space
 $f \in C_0(M)$, $K = \text{supp } f$.

Claim: $\exists \xi_i \in C_0(M)$, $0 \leq \xi_i \leq 1$, $i=1, \dots, N$ s.t.

① $\text{supp } \xi_i \subset \Omega_i$, Ω_i is a coordinate neighbourhood.

② $\sum_{i=1}^N \xi_i = 1$ in K .

$$f = \sum \xi_i f \quad \text{supp } \xi_i f \subset \Omega_i$$

$$\text{Def. } I(f) = \sum_{i=1}^N I_{\Omega_i}(\xi_i f|_{\Omega_i})$$

Claim: The definition is independent of $\{\xi_i\}$.

Step 3. $f: M \rightarrow \mathbb{R}$.

THM (Riesz representation) Let X be a locally compact Hausdorff space. I is a nonnegative linear functional on $C_c(X)$, $\exists!$ regular Borel measure μ on X , such that.

$$I(f) = \int f \, d\mu \quad \text{for all } f \in C_0(X)$$

$I : C_0(M) \rightarrow \mathbb{R} \iff$ regular Borel measure ν .

$$I(f) = \int_M f \, d\nu \quad f \in C_0(M)$$

$\nu \rightsquigarrow$ volume measure. $A \subset M$: ν -measure. $\nu(A)$ volume of A

$f : M \rightarrow \mathbb{R}$. ν -measurable.

$$f \geq 0, \quad \int_M |f| \, d\nu < +\infty$$

$\int_M f \, d\nu$ volume integral of f .

$X \in \mathcal{V}(M)$.

$\Omega \subset \mathbb{R}^n$ open $F \in C^\infty(\Omega, \mathbb{R}^n)$, $x_0 \in \Omega$, $v \in T_{x_0}\Omega$.

$$D_v F(x_0) = \lim_{t \rightarrow 0} \frac{F(x_0 + vt) - F(x_0)}{t}$$

$F \in \mathcal{V}(M)$

Def : Let $D : \mathcal{V}(M) \times \mathcal{V}(M) \rightarrow \mathcal{V}(M)$.

$$\text{If } ① \quad D_X(Y+Z) = D_X Y + D_X Z \quad \forall f, g \in C^\infty(M)$$

$$②. \quad D_X(fY) = Xf \cdot Y + f D_X Y \quad X, Y, Z \in \mathcal{V}(M).$$

$$③ \quad D_{fX+gY} Z = f D_X Z + g D_Y Z$$

We say D is an affine connection on M .

$$F : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$(D_X F)_p \stackrel{\Delta}{=} D_{X(p)} F \quad X, Y \in \mathcal{V}(\Omega)$$

D : affine connection on differential manifold.

Claim. Let $X, \hat{X}, Y, \tilde{Y} \in \mathcal{V}(M)$, $p \in M$.

① If $X(p) = \hat{X}(p)$ then

$$(D_X Y)_p = (D_{\hat{X}} Y)_p.$$

② If $\tilde{Y} \equiv Y$ in some neighbourhood of p , then

$$(D_X Y)(p) = (D_X \tilde{Y})(p)$$

Proof ① since $X \mapsto D_X Y$ is $\mathcal{F}$ -linear

②. Let U be a neighbourhood of p such that

$$Y = \tilde{Y} \quad \text{in } U.$$

Let $\psi \in C^\infty(M)$, $\text{supp}(\psi) \subset U$, then

$$Y - \tilde{Y} = \psi(Y - \tilde{Y}) \quad \text{in } M.$$

then

$$D_X(Y - \tilde{Y})(p) = (D_X \psi(Y - \tilde{Y}))(p) = \psi(p) (D_X(Y - \tilde{Y}))(p) + X\psi(p) (Y - \tilde{Y})(p) = 0.$$

Def. Let $\Omega \subset M$ be open, $p \in \Omega$, $v \in T_p \Omega$, $X \in \mathcal{V}(\Omega)$. Def.

$$D_v X = (D_V \tilde{X})(p)$$

where $V, \tilde{X} \in \mathcal{V}(M)$, $V(p) = v$, $\tilde{X} = X$ in some neighbourhood of p .

Prop. $D_v(\alpha X + \beta Y) = \alpha D_v X + \beta D_v Y$. $X, Y \in \mathcal{V}(\Omega)$

$$D_v(fX) = \langle v, f \rangle X(p) + D_v X f(p). \quad \text{v, w} \in T_p M$$

$$D_{\alpha v + \beta w} X = \alpha D_v X + \beta D_w X.$$

Def. Let $\Omega \subset M$ be open, $X, Y \in \mathcal{V}(\Omega)$. Def

$$(D_X Y)(p) = D_{X(p)} Y \quad p \in \Omega.$$

$$D_X Y \in \mathcal{V}(\Omega).$$

$$D: \mathcal{V}(\Omega) \times \mathcal{V}(\Omega) \rightarrow \mathcal{V}(\Omega)$$

affine connection on Ω .

Ω : coordinate neighbourhood.

φ : coordinate on Ω .

$$\frac{\partial}{\partial x^i} \Big|_p = d\varphi \left(\frac{\partial}{\partial x^i} \Big|_{\hat{p}} \right).$$

$$D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k} \quad \Gamma_{ij}^k \leadsto \text{connection coefficients.}$$

$$X = X^i \frac{\partial}{\partial x^i} \quad D_X Y = D_{X^i \frac{\partial}{\partial x^i}} Y^j \frac{\partial}{\partial x^j}$$

$$Y = Y^i \frac{\partial}{\partial x^i}$$

$$= X^j D_{\frac{\partial}{\partial x^j}} (Y^i \frac{\partial}{\partial x^i})$$

$$= X^j \left(Y^i \Gamma_{ij}^k \frac{\partial}{\partial x^k} + \frac{\partial Y^i}{\partial x^j} \frac{\partial}{\partial x^i} \right)$$

$$= X^j \left(Y^i \Gamma_{ij}^k + \frac{\partial Y^i}{\partial x^j} \right) \frac{\partial}{\partial x^k}$$

$$\Omega \subset \mathbb{R}^n, X, Y \in \mathcal{V}(\Omega).$$

$$D_X Y = (D_X Y^k) e_k = X^i D_i Y^k e_k.$$

D affine connect on M $\gamma : (a, b) \rightarrow M$ smooth

$$\gamma(t_0) = p \quad \gamma'(t_0) = v$$

Let $X \in \mathcal{V}(M)$. Then $D_v X$ depend only on $X \circ \gamma$

Proof. $D_v X = D_v (X^i \frac{\partial}{\partial x^i})$

$$= v(X^i) \frac{\partial}{\partial x^i} |_p + X^i(p) D_v \frac{\partial}{\partial x^i}$$

$$= \frac{d}{dt} \Big|_{t_0} X^i \circ \gamma \frac{\partial}{\partial x^i} |_p + X^i \circ \gamma(t_0) D_{\gamma'(t_0)} \frac{\partial}{\partial x^i}$$

Def: Let $\gamma : (a, b) \rightarrow M$, $X : (a, b) \rightarrow TM$ be smooth. If

$$X(t) \in T_{\gamma(t)} M \quad t \in (a, b)$$

X is called a smooth vector field along γ .

$\mathcal{V}(\gamma)$ smooth fields along γ

Let $f, g \in C^\infty((a, b))$, $X, Y \in \mathcal{V}(\gamma)$

$$(fX + gY)(t) = f(t) Y(t) + g(t) Y(t).$$

Then. $fX + gY \in \mathcal{V}(\gamma)$

Example

①. $\gamma' \in \mathcal{V}(\gamma)$

②. $X \in \mathcal{V}(M)$ $X \circ \gamma \in \mathcal{V}(\gamma)$

③. $f \in C^\infty(a, b)$, $p \in M$ $v \in T_p M$, $v \neq 0$.

$$X(t) = f(t)v. \quad \leadsto \text{In this case, } \gamma(t) \equiv p$$

If $X = X(t) \in \mathcal{V}(\gamma)$, we define:

$$\left. \frac{dX}{dt} \right|_{t_0} = D_{\gamma'(t_0)} \tilde{X}$$

where $\tilde{X} \in \mathcal{V}(M)$ and $\tilde{X} \circ \gamma = X$.

Prop. Let D be an affine connection on M , Then $\exists !$ operation.

$$\frac{D}{dt} : \mathcal{V}(\gamma) \rightarrow \mathcal{V}(\gamma)$$

such that ① $\frac{D}{dt}(X+Y) = \frac{DX}{dt} + \frac{DY}{dt}$

②. $\frac{D}{dt}(fX) = f'X + f \frac{DX}{dt}$, $X, Y \in \mathcal{V}(\gamma)$, $f \in C^\infty(a, b)$

③. Let $X \in \mathcal{V}(\gamma)$. If $\exists$ a smooth field $\tilde{X}$ define in some neighbourhood of $p = \gamma(t_0)$ such that

$$\tilde{X} \circ \gamma = X \quad \text{in some neighbourhood of } t_0.$$

then

$$\left. \frac{DX}{dt} \right|_{t_0} = D_{\gamma'(t_0)} \tilde{X}$$

Proof.

Let $X = X^i \frac{\partial}{\partial x^i} \in \mathcal{V}(\gamma)$ $(X^i = X^i(t), \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} \circ \gamma(t))$.

Note that:

$$\frac{dX}{dt} = \left(\frac{dX^k}{dt} + \Gamma_{ij}^k \frac{dX^i}{dt} X^j \right) \frac{\partial}{\partial x^k} \leadsto \text{define } \frac{d}{dt} = \frac{D}{dt}.$$

Indeed:

$$\begin{aligned} \frac{dX}{dt} &= \frac{d}{dt} \left(X^i \frac{\partial}{\partial x^i} \right) = \frac{dX^i}{dt} \frac{\partial}{\partial x^i} + X^i \frac{d}{dt} \left(\frac{\partial}{\partial x^i} \circ \gamma \right) \leadsto \frac{\partial}{\partial x^i} \in \mathcal{V}(M) \\ &= \frac{dX^i}{dt} \frac{\partial}{\partial x^i} + X^i D_{\gamma'} \frac{\partial}{\partial x^i} \\ &= \frac{dX^i}{dt} \frac{\partial}{\partial x^i} + X^j D_{\gamma'} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} \\ &= \left(\frac{dX^k}{dt} + X^j \frac{dX^i}{dt} \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k} \end{aligned}$$

$$\textcircled{1} \gamma' \in \mathcal{V}(\gamma) \quad \gamma' = \frac{dx^i}{dt} \frac{\partial}{\partial x^i}$$

$$\frac{D}{dt} \frac{d\gamma}{dt} = \frac{D}{dt} \frac{dx^i}{dt} \frac{\partial}{\partial x^i} = \frac{d^2 x^i}{dt^2} \frac{\partial}{\partial x^i} + \frac{dx^i}{dt} \frac{D}{dt} \frac{\partial}{\partial x^i}$$

$$= \left(\frac{d^2 \gamma^k}{dt^2} + \frac{dx^i}{dt} \frac{d\gamma^j}{dt} \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k} \quad \gamma: \text{加速度.}$$

$\frac{D}{dt} \frac{\partial}{\partial x^i} = \frac{D}{dt} \left(\frac{\partial}{\partial x^i} \gamma \right) = \nabla_{\gamma} \frac{\partial}{\partial x^i}$

$$\textcircled{2} X(t) = f(t)v = f(t)x^i \frac{\partial}{\partial x^i} \quad f \text{ smooth, } v \in T_p M. \quad \text{In this case } \gamma \equiv p \text{ and thus } \gamma' = 0.$$

$$\frac{DX}{dt} = \frac{df}{dt} v^k \frac{\partial}{\partial x^k} = f'(t)v$$

Let (M, g) Riemann manifold, take $M = \mathbb{R}^n$, then.

$$D_X Y = X(Y^j) \frac{\partial}{\partial x^j} = x^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j}$$

$$\cdot [X, Y] = D_X Y - D_Y X$$

$$\cdot X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle$$

Def. Let D be an affine connection on M . If

$$X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle \quad \forall X, Y, Z \in \mathcal{V}(M)$$

We say D is a metric connection.

Claim: D is a metric connection, $v \in T_p M$.

$$v \langle Y, Z \rangle = \langle D_v Y, Z(p) \rangle + \langle Y(p), D_v Z \rangle \quad Y, Z \in \mathcal{V}(M).$$

Prop 1. D is a metric connection, $X, Y \in \mathcal{V}(\gamma)$, Then

$$\frac{d}{dt} \langle X, Y \rangle = \left\langle \frac{D}{dt} X, Y \right\rangle + \left\langle X, \frac{D}{dt} Y \right\rangle.$$

Proof: $\forall X = x^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j} \in \mathcal{V}(\gamma)$, then.

$$\langle X, Y \rangle = x^i Y^j \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$$

It follows that

$$\frac{d}{dt} \langle X, Y \rangle = \left(\frac{dx^i}{dt} Y^j + \frac{dY^j}{dt} x^i \right) \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle + x^i Y^j \frac{d}{dt} \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$$

$$\text{But } \frac{d}{dt} \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \frac{d}{dt} \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \circ \gamma(t) = \left\langle \gamma'(t), \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \right\rangle$$

$$= \left\langle D_{Y^{(1)}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle + \left\langle \frac{\partial}{\partial x^i}, D_{Y^{(1)}} \frac{\partial}{\partial x^j} \right\rangle$$

$$= \left\langle \frac{D}{dx} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle + \left\langle \frac{\partial}{\partial x^i}, \frac{D}{dx} \frac{\partial}{\partial x^j} \right\rangle.$$

Def. D is an affine connection on M if.

$$D_X Y - D_Y X = [X, Y]. \quad \forall X, Y \in \mathcal{V}(M).$$

We say D is symmetric. torsion free.

Remark. D symmetric $\Rightarrow D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$

$$\Leftrightarrow \Gamma_{ij}^k = \Gamma_{ji}^k$$

$$T : \mathcal{V}(M) \times \mathcal{V}(M) \rightarrow \mathcal{V}(M). \quad \text{torsion of } D.$$

$$(X, Y) \mapsto D_X Y - D_Y X - [X, Y].$$

Def. Let (M, g) be a Riemann manifold. Let D be an affine connection on M . If D is a symmetric metric connection. we say D is a Riemann connection.

Example: (Riemann connection on S^n) There is a unique Riemann connection on S^n .

$$\nabla_X Y =: (\bar{\nabla}_X \bar{Y})^T \quad \forall X, Y \in \mathcal{V}(S^n)$$

where $\bar{\nabla}$ is the Riemann connection on $\mathbb{R}^{n+1}$ and.

$$\bar{X}|_{S^n} = X, \quad \bar{Y}|_{S^n} = Y$$

Proof: $\vec{n}(p)$ normal vector at p in S^n , then let.

$$\nabla_X Y = \bar{\nabla}_X \bar{Y} - \langle \bar{\nabla}_X \bar{Y}, \vec{n} \rangle \vec{n}$$

Clearly, ∇ is a connection on S^n .

• symmetric: $\nabla_X Y - \nabla_Y X = [\bar{X}, \bar{Y}] - \langle [\bar{X}, \bar{Y}], \vec{n} \rangle \vec{n}$

But $X, Y \in TS^n \Rightarrow [X, Y] \in TS^n \Rightarrow [\bar{X}, \bar{Y}] \in TS^n$

$$\Rightarrow \langle [\bar{X}, \bar{Y}], \vec{n} \rangle = 0.$$

• metric: $\langle \nabla_X Y, Z \rangle = \langle \bar{\nabla}_X \bar{Y}, Z \rangle - \langle \bar{\nabla}_X \bar{Y}, \vec{n} \rangle \langle \vec{n}, Z \rangle$

$$= \langle \bar{\nabla}_X \bar{Y}, Z \rangle$$

and thus $\langle \nabla_X Y, Z \rangle + \langle \nabla_X Y, Z \rangle = \bar{X} \langle \bar{Y}, \bar{Z} \rangle = X \langle Y, Z \rangle$

Notions:

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \quad g^{ij} = (g_{ij})^{-1}$$

Prop. $\Gamma_{ij}^k = \frac{1}{2} \left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right) g^{mk}$

$\downarrow$

$\Gamma_{ij,m}$

Remark: It follows that the Riemann connection is unique determined by Riemann metric!

Proof. $\frac{\partial g_{im}}{\partial x^j} = \frac{\partial}{\partial x^j} \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^m} \right\rangle = \left\langle D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^m} \right\rangle + \left\langle \frac{\partial}{\partial x^i}, D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^m} \right\rangle$ ①

$$\frac{\partial g_{jm}}{\partial x^i} = \frac{\partial}{\partial x^i} \left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^m} \right\rangle = \left\langle D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^m} \right\rangle + \left\langle \frac{\partial}{\partial x^j}, D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^m} \right\rangle$$
 ②

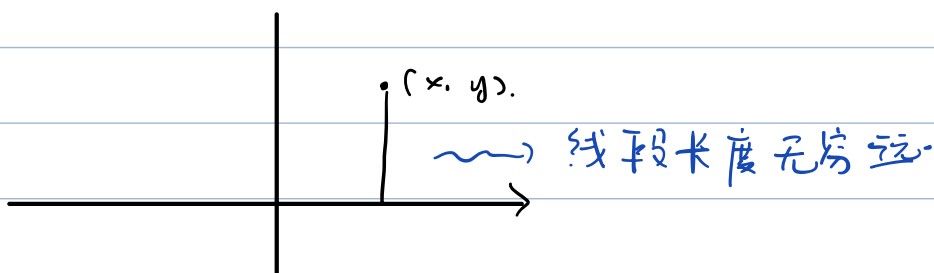
$$\frac{\partial g_{ij}}{\partial x^m} = \frac{\partial}{\partial x^m} \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \left\langle D_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle + \left\langle \frac{\partial}{\partial x^i}, D_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^j} \right\rangle$$
 ③

$$\textcircled{1} + \textcircled{2} - \textcircled{3}$$

$$\begin{aligned} \frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} &= 2 \left\langle D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^m} \right\rangle = 2 \left\langle \Gamma_{ij}^k \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m} \right\rangle \\ &= 2 \Gamma_{ij}^l g_{lm} \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right) g^{mk} &= \Gamma_{ij}^l g_{lm} g^{mk} \\ &= \Gamma_{ij}^l \delta_l^k \\ &= \Gamma_{ij}^k \end{aligned}$$

$$H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\} \quad I: \mathbb{R}_+^2 \rightarrow H^2, \\ g_{ij}(x, y) = \frac{\delta_{ij}}{y^2} \quad \text{罗曼切夫斯基平面}$$



$$\Gamma_{ij}^k = \frac{1}{2} \left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right) g^{mk}.$$

$$G = \begin{bmatrix} \frac{1}{y^2} & \\ & \frac{1}{y^2} \end{bmatrix} \quad G^{-1} = \begin{bmatrix} y^2 & \\ & y^2 \end{bmatrix} \\ g^{mk} = y^2 \delta_m^k.$$

$$\Gamma_{11}^1 = \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial x} \right) g^{11} + \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial y} \right) g^{21} \\ = 0.$$

$$\Gamma_{11}^2 = \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial x} \right) g^{12} + \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial y} \right) g^{22} \\ = -\frac{1}{2} \left(\frac{1}{y^2} \right)' \cdot y^2 \\ = -\frac{1}{y}.$$

$$\Gamma_{12}^1 = \frac{1}{2} \left(\frac{\partial g_{11}}{\partial y} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial x} \right) g^{11} + \frac{1}{2} \left(\frac{\partial g_{11}}{\partial y} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial y} \right) g^{21} \\ = \frac{1}{2} \frac{\partial g_{11}}{\partial y} \cdot g^{11} = \\ = -\frac{1}{y}.$$

$$\Gamma_{12}^2 = \frac{1}{2} \left(\frac{\partial g_{11}}{\partial y} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial x} \right) g^{12} + \frac{1}{2} \left(\frac{\partial g_{11}}{\partial y} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial y} \right) g^{22} \\ = 0$$

$$\Gamma_{21}^1 = \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x} + \frac{\partial g_{11}}{\partial y} - \frac{\partial g_{11}}{\partial x} \right) g^{11} + \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x} + \frac{\partial g_{11}}{\partial y} - \frac{\partial g_{11}}{\partial y} \right) g^{12} \\ = -\frac{1}{y}$$

$$\Gamma_{21}^1 = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right) g^{12} + \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \right) g^{22} = 0.$$

Vector field along a surface.

$$f = f(s, t) \quad (a, b) \times (c, d) \rightarrow M \quad \text{surface}$$

$$X = X(s, t) \quad (a, b) \times (c, d) \rightarrow TM.$$

$$X(s, t) \in T_{f(s, t)} M.$$

X : vector field along f .

$$F: A \times B \rightarrow C. \quad F = F(s, t). \quad F_s(t) = F_t(s) = F(s, t).$$

$f_s, f_t \rightsquigarrow$ two families of curves.

$$\frac{\partial f}{\partial t} = \frac{df}{dt}. \quad \frac{\partial f}{\partial s} = \frac{df}{ds} \rightsquigarrow \text{vector field along } f.$$

$$\frac{\partial X_s}{\partial t} = \frac{d}{dt} X_s. \quad \frac{\partial X_t}{\partial s} = \frac{d}{ds} X_t \rightsquigarrow \text{vector field along surface.}$$

$$\frac{\partial^2 f}{\partial t^2}, \frac{\partial^2 f}{\partial s^2}, \frac{\partial^2 f}{\partial t \partial s}, \frac{\partial^2 f}{\partial s \partial t}$$

$$\text{Prop: } \frac{\partial^2 f}{\partial s \partial t} = \frac{\partial^2 f}{\partial t \partial s} \quad f: \mathbb{R}^2 \rightarrow M.$$

$$(U, \varphi) \quad \text{local chart.} \quad \frac{\partial}{\partial x^i} = d\varphi^{-1} \left(\frac{\partial}{\partial x^i} \right)$$

$$f^i = \varphi^i \circ f \quad X = X^i \frac{\partial}{\partial x^i}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f^i}{\partial s} \frac{\partial}{\partial x^i}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f^i}{\partial t} \frac{\partial}{\partial x^i}$$

$$\frac{\partial X}{\partial s} = \left(\frac{\partial X^k}{\partial s} + \Gamma_{ij}^k \frac{\partial f^i}{\partial s} X^j \right) \frac{\partial}{\partial x^k}$$

$$\frac{\partial}{\partial s} \frac{\partial f}{\partial t} = \left(\frac{\partial}{\partial s} \frac{\partial f^k}{\partial t} + \Gamma_{ij}^k \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial t} \right) \frac{\partial}{\partial x^k}$$

$$\frac{\partial}{\partial t} \frac{\partial f}{\partial s} = \left(\frac{\partial}{\partial t} \frac{\partial f^k}{\partial s} + \Gamma_{ij}^k \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial s} \right) \frac{\partial}{\partial x^k}$$

$$\frac{\partial^2 X}{\partial s \partial t} - \frac{\partial^2 X}{\partial t \partial s} \rightsquigarrow \text{curvature of } M.$$

$\dim M = \dim N$. F : bijective. F, F^{-1} smooth.

$F: M \rightarrow N$ smooth. local diffeomorphism. if:

$\forall p \in M, \exists U \in \mathcal{N}(p), V \in \mathcal{N}(F(p))$. Such that.

$$F: U \rightarrow V$$

is a diffeomorphism.

Prop: Let $F: M \rightarrow N$ be smooth

F is a local diffeomorphism $\Leftrightarrow \forall p \in M, dF_p$ is bijective.

(Inverse function theorem)

$\dim M = \dim N$ Riemann manifold.

$F: M \rightarrow N$ diffeomorphism. If

$$\forall u, v \in T_p M.$$

$$\langle dF_p(u), dF_p(v) \rangle_{F(p)} = \langle u, v \rangle_p.$$

$$\forall p \in M.$$

We say F : Riemann isometry.

$F: M \rightarrow N$ smooth. If $\forall p \in M, \exists U \in \mathcal{N}(p), V \in \mathcal{N}(F(p))$ such that.

$$F: U \rightarrow V$$

is a Riemann isometry. F : locally Riemann isometry.

Prop. $F: M \rightarrow N$ smooth.

F is local Riemann isometry

$$\Leftrightarrow \langle dF_p(u), dF_p(v) \rangle = \langle u, v \rangle \quad \forall p \in M, u, v \in T_p M.$$

$F: \mathbb{R} \longrightarrow S^1$ local Riemann isometry.
 $\theta \longmapsto (\cos \theta, \sin \theta)$

$$dF_p\left(\frac{d}{d\theta}\right) = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}$$

$$\begin{aligned} \langle dF_p\left(\frac{d}{d\theta}\right), dF_p\left(\frac{d}{d\theta}\right) \rangle &= \sin^2 \theta + \cos^2 \theta = 1 \\ &= \left\langle \frac{d}{d\theta}, \frac{d}{d\theta} \right\rangle \end{aligned}$$

Prop. $F: M \longrightarrow N$ local Riemann isometry.

$$\textcircled{1}. L(F(\gamma)) = L(\gamma)$$

$$\textcircled{2}. d_N(F(p), F(q)) \leq d_M(p, q) \quad p, q \in M$$

(复叠映照).

Def 1 Let $\gamma: (a, b) \longrightarrow M$ be smooth. The equation.

$$\frac{d^2 \gamma}{dt^2} = 0.$$

is called the geodesic equation.

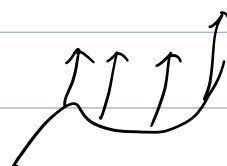
$$\frac{d^2 \gamma^k}{dt^2} + \Gamma_{ij}^k \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0. \quad k=1, \dots, n.$$

$\gamma \equiv p$ 为一个解.

In $\mathbb{R}^n$:

$$(\gamma(t) = at + b \quad a \neq 0.)$$

$\frac{d}{dt} \gamma$ velocity.



$$0 = \frac{dX}{dt} \rightsquigarrow X \text{ parallel}$$

$$0 = \frac{d^2 \gamma}{dt^2} \rightsquigarrow \frac{d\gamma}{dt} \text{ parallel}$$

Def. If γ satisfies geodesic equation, and $\gamma \neq p \forall p \in M$.
 γ : geodesic.

Let $\gamma: [a, b] \rightarrow M$. If $\exists$ a geodesic $\tilde{\gamma}: [a', b'] \rightarrow M$ such that.

$$\tilde{\gamma}|_{[a, b]} = \gamma$$

γ is called a geodesic segment.

Prop. If γ is a geodesic, then

$$\alpha(t) = \gamma(kt+b), \quad k, b \in \mathbb{R}, k \neq 0.$$

is a geodesic.

线性重参数化.

Proof.

$$\begin{aligned} \frac{d^2 \alpha}{dt^2} &= \frac{d}{dt} \left(\dot{\alpha}^i \frac{\partial}{\partial x^i} \right) = \left(\ddot{\alpha}^2 + \Gamma_{ij}^k \frac{d\alpha^i}{dt} \frac{d\alpha^j}{dt} \right) \frac{\partial}{\partial x^k} \\ &= k' \left(\ddot{\gamma}(kt+b) + \Gamma_{ij}^k \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \right) \frac{\partial}{\partial x^k} = 0 \end{aligned}$$

Prop. If γ is a geodesic, then $|\gamma'| = c$.

Proof.

$$\frac{d}{dt} |\gamma'|^2 = \frac{d}{dt} \left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = 2 \left\langle \frac{d^2 \gamma}{dt^2}, \frac{d\gamma}{dt} \right\rangle = 0$$

$$\text{i.e. } |\gamma'|^2 = c$$

If $|\gamma'| = c_0$, $\alpha(t) = \gamma(\frac{1}{c_0}t + b)$, then

$$\alpha' = \frac{1}{c_0} \gamma' \quad |\alpha'| = 1$$

Initial problem.

$$(1) \begin{cases} \gamma'' = 0 \\ \gamma'(0) = v. \end{cases} \quad v \in TM \quad \Leftrightarrow \quad \begin{cases} Y'' = 0 \\ Y(0) = v \quad Y(1) = \gamma. \end{cases} \quad v \in T_p M$$

local $\exists!$ $\leadsto$ maximal solution. $(a_v, b_v) \longrightarrow \mathcal{M}$.

$$-\infty \leq a_v < 0 < b_v \leq +\infty$$

γ depends smoothly on v and t .

S^2 上测地线 $\leadsto$ 速度为常数的大圆.

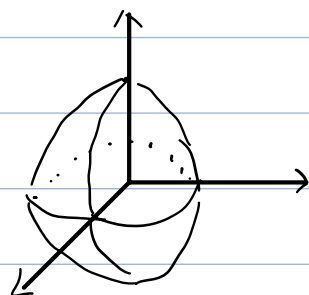
$$\forall v = x_0 \frac{\partial}{\partial x} + y_0 \frac{\partial}{\partial y} + z_0 \frac{\partial}{\partial z} \in TM$$

$$\text{where } x_0^2 + y_0^2 + z_0^2 = 1.$$

Let $u, v \in \mathbb{R}^3$, $|u| = |v| = 1$ $u \perp v$.

$$\gamma(t) = \cos(k+t)b u + \sin(k+t)b v.$$

$$\begin{aligned} \begin{cases} \gamma'(t) = -k \sin(k+t)b u + k \cos(k+t)b v \\ \gamma''(t) = (-k^2 \cos(k+t)b u - k^2 \sin(k+t)b v)^T \\ = \gamma''(t) - (\gamma(t), \gamma''(t)) \gamma''(t). \end{cases} \end{aligned}$$



$\Gamma(v, t)$ solution of 1)

$$\Omega = \{(v, t) : v \in TM, t \in (a_v, b_v)\}.$$

$$\Omega \subset TM \times \mathbb{R} \text{ open. } \{0_p\} \times \mathbb{R} \in \Omega.$$

$\Gamma: \Omega \longrightarrow \mathcal{M}$ smooth.

Prop. $\Gamma(kv, t) = \Gamma(v, kt)$ $k \in \mathbb{R}$. 齐性.

$$\begin{aligned} \text{Proof. } \frac{d}{dt} \big|_0 \Gamma(v, kt) &= \frac{d}{dt} \big|_0 \Gamma'(v, kt) \frac{\partial}{\partial x_i} \\ &= k \Gamma'(v, 0) \frac{\partial}{\partial x_i} \\ &= kv \end{aligned}$$

Define

$$D = \{v \in TM : 1 \in (a_v, b_v)\}$$

$$\exp : D \rightarrow M$$

exponential map.

$$v \mapsto P(v, 1)$$

Prop: D is open $0_p \in D$. $\exp$ is smooth.

$t \mapsto \exp(tv)$ $v \neq 0$ geodesic with initial velocity v .

$D_p = D \cap T_p M \rightsquigarrow$ neighbourhood of 0_p star-shaped
 $\exp_p(v) = \exp(v)$ $v \in T_p M$.

$$v \in D_p \Rightarrow tv \in D_p \text{ (act.)}$$

$$\exp_p : D_p \subset T_p M \rightarrow M. \quad v \in D_p$$

$$d(\exp_p)_v : T_p D_p \rightarrow T_v M.$$

linear function.

$$\hookrightarrow T_p M.$$

$$d(\exp_p)_0 : T_p M \rightarrow T_p M.$$

Prop $d(\exp_p)_0 = \mathbb{1}_{T_p M}.$

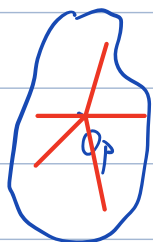
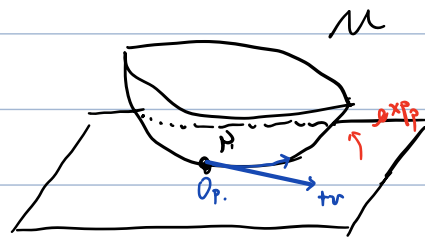
Proof.

$$d(\exp_p)_0 v = \frac{d}{dt} \exp_p(tv) = v.$$

$\hookrightarrow \exists$ some neighbourhood U of 0_p such that.

$$\exp_p : U \rightarrow \exp_p(U)$$

is a diffeomorphism. $\exp_p(U)$ geodesic neighbourhood of p .



$U \subset T_p M$

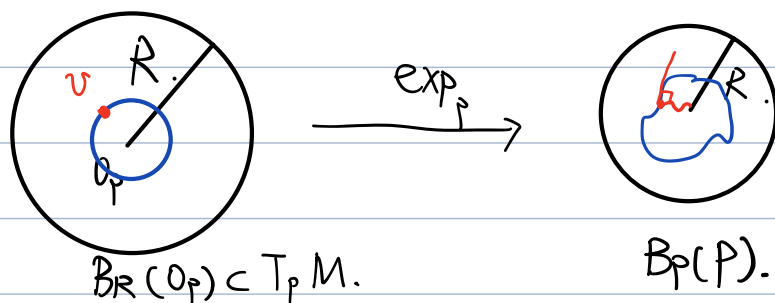
$$\xrightarrow{\exp_p}$$



geodesic ball.

$$\text{If } U = B_R(0_p) \subset T_p M, \quad B_R(p) = \exp_p(B_R(0_p)).$$

Assume $B_R(p)$ is a geodesic ball.



半径与球面垂直?

Gauss lemma.

$$B_R(p) = \{q \in M : d(p, q) < R\}$$

$$0 < r < R$$

$$\exp_p(\partial B_r(0_p)) = S_r(p) \quad \text{geodesic sphere.}$$

$$|v| = r.$$

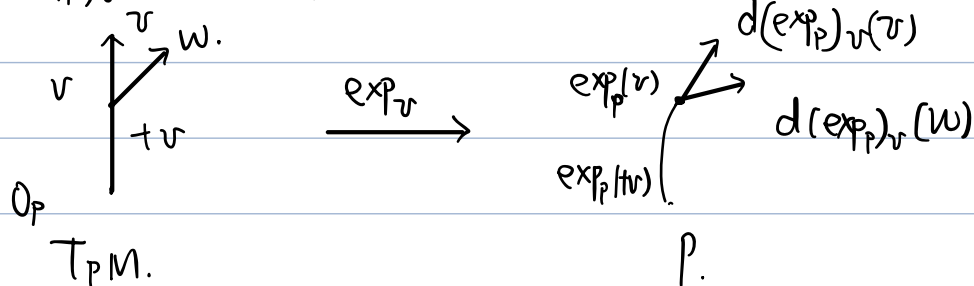
$$\exp_p(tv) \quad (0 \leq t \leq 1) \quad \text{radial geodesic.}$$

Prop. Let $p \in M$, $\exists$ some neighborhood U of p and $r > 0$ such that $\forall q \in U$, $\exp_q : B_r(0_q) \rightarrow \exp_q(B_r(0_q))$ is a diffeomorphism and $B_r(0_q) \supset U$.

$U \rightsquigarrow$ total geodesic neighborhood.

Theorem Let $p \in M$, $v \in \text{Dom}(\exp_p)$, $w \in T_p M$. Then.

$$\langle d(\exp_p)_v(v), d(\exp_p)_v(w) \rangle = \langle v, w \rangle.$$



Corollary

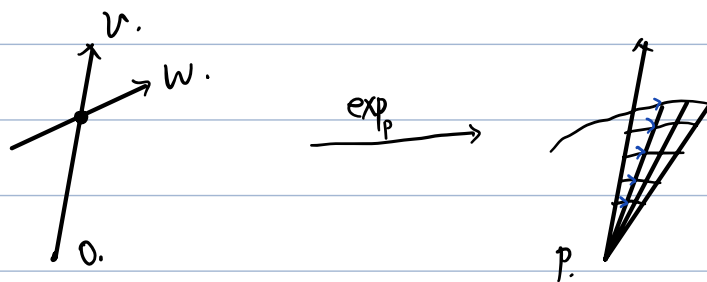
$$u \parallel v$$

$$w \perp v$$

$$\langle d(\exp_p)(u), d(\exp_p)(w) \rangle = \langle u, w \rangle$$

$$\langle d(\exp_p)(u), d(\exp_p)(w) \rangle = 0.$$

Proof of thm.



Let $\alpha : (-\varepsilon, \varepsilon) \rightarrow \text{Dom } \exp_p$

$$\alpha(0) = v, \quad \alpha'(0) = w.$$

Let $f(s, t) = \exp_p(t\alpha(s))$

$$\frac{\partial f}{\partial s} = d(\exp_p)_{t\alpha(s)}(t\alpha'(s)) \quad \frac{\partial f}{\partial t} = d(\exp_p)_{t\alpha(s)}(\alpha(s)).$$

$$\frac{\partial f}{\partial s}(0, 1) = d(\exp_p)_v(w) \quad \frac{\partial f}{\partial t}(0, 1) = d(\exp_p)_v(v)$$

It suffices to check

$$\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle \Big|_{t=1, s=0} = \langle v, w \rangle$$

Indeed:

$$\begin{aligned} \frac{\partial}{\partial t} \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle &= \left\langle \frac{\partial}{\partial t} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{\partial f}{\partial s}, \frac{\partial^2 f}{\partial t^2} \right\rangle \\ &= \left\langle \frac{\partial}{\partial t} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle = \left\langle \frac{\partial}{\partial s} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle \\ &= \frac{1}{2} \frac{\partial}{\partial s} \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle \\ &= \frac{1}{2} \frac{\partial}{\partial s} \left| \frac{\partial f}{\partial t} \right|^2 = \frac{1}{2} \frac{\partial}{\partial s} |\alpha(s)|^2 \\ &= \langle \alpha'(s), \alpha(s) \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle &= \langle \alpha'(s), \alpha(s) \rangle t \\ \Rightarrow \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle (0, 1) &= \langle w, v \rangle \end{aligned}$$

Def. Let $\gamma : [a, b] \rightarrow M$ be piecewise smooth. If

$$L(\gamma) = d(\gamma(a), \gamma(b))$$

We say γ is minimizing.

Theorem. Let $p \in M$, $B_R(p)$ be a geodesic ball, $v \in B(0_p, R) \setminus \{0_p\}$, $q = \exp_p(v)$

Let $\alpha: [0, 1] \rightarrow M$ be a piecewise smooth curve with.

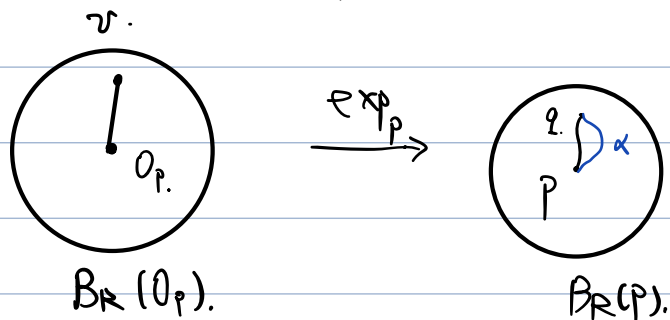
$$\alpha(0) = p$$

$$\alpha(1) = q.$$

Then. ① $L(\alpha) \geq |v|$

② if $L(\alpha) = |v|$ and $|\alpha'| \leq C$, then.

$$\alpha(t) = \exp_p(tv) \quad t \in [0, 1]$$



Proof. Assume ①. $\alpha(t) \in B_R(p)$. $t \in [0, 1]$.

$$\textcircled{2}. \alpha(t) \neq p \quad \forall t \in (0, 1]$$

Let $\beta = (\exp)^{-1} \circ \alpha$, then β is piecewise smooth.

$$\beta(0) = 0_p.$$

$$\beta(1) = v.$$

$$\alpha = \exp_p(\beta).$$

$$\beta \neq 0 \quad \forall t \in (0, 1]$$

$$\begin{cases} (\beta')^\perp = \langle \beta', \hat{\beta} \rangle \hat{\beta} \\ (\beta')^T = \beta' - (\beta')^\perp \end{cases} \quad \hat{\beta} = \frac{\beta}{|\beta|}$$

$$\Rightarrow \begin{cases} (\alpha')^\perp = (d\exp_p)_\beta((\beta')^\perp) \\ (\alpha')^T = (d\exp_p)_\beta((\beta')^T) \end{cases} \quad [\text{Gauss Lemma}]$$

$$\alpha' = (\alpha')^\perp + (\alpha')^T. \quad \text{切向, 无做功}$$

$$\begin{aligned} \hookrightarrow |\alpha'| &= \sqrt{|(\alpha')^\perp|^2 + |(\alpha')^T|^2} \geq |(\alpha')^\perp| = |(\beta')^\perp| = |\langle \beta', \hat{\beta} \rangle| \\ &= \left| \frac{\langle \beta', \beta \rangle}{|\beta|} \right| = \frac{1}{2|\beta|} \left| \frac{d}{dt} |\beta|^2 \right| = \left| \frac{d}{dt} |\beta| \right| \end{aligned}$$

$$\begin{aligned} L(\alpha) &= \int_0^1 |\alpha'| dt \geq \int_0^1 \left| \frac{d}{dt} |\beta| \right| dt \geq \int_0^1 \frac{d}{dt} |\beta| dt \\ &= |\beta(1)| - |\beta(0)| = |v|. \end{aligned}$$

Now, assume $L(\alpha) = |v|$, $|\alpha'| \leq C$

Claim 1: $|\beta'| = c$, $L(\beta) = |v|$.

$$L(\alpha) = 0 \Rightarrow (\alpha')^T = 0 \Rightarrow (\beta')^T = 0.$$

$$c = |\alpha'| = |d(\exp_p)_\beta(\beta')| = |d(\exp_p)_\beta(\beta')^T| \\ = |(\beta')^T| = |\beta'|$$

$$\Rightarrow L(\alpha) = \int_0^1 |\alpha'| = \int_0^1 |\beta'| = L(\beta).$$

Claim 2: $\beta = tv$.

$$\begin{cases} \frac{d}{dt} |\beta'|^2 = 2 \langle \beta', \beta'' \rangle = 0. \\ \beta' = (\beta')^\perp + (\beta')^T = (\beta')^\perp = \langle \beta', \hat{\beta} \rangle \hat{\beta} = \frac{\langle \beta', \beta \rangle \beta}{|\beta|^2}. \\ \langle \beta'', \hat{\beta} \rangle \langle \beta', \hat{\beta} \rangle = 0 \\ \Rightarrow \langle \beta'', \beta \rangle = 0. \end{cases}$$

$$\beta'' = \frac{(\langle \beta'', \beta \rangle + \langle \beta', \beta' \rangle) \beta + \langle \beta', \beta \rangle \beta' - 2 \langle \beta', \beta \rangle^2 \beta}{|\beta|^4} \parallel \beta.$$

$$\Rightarrow \beta'' = 0.$$

$$\Rightarrow \beta = tv$$

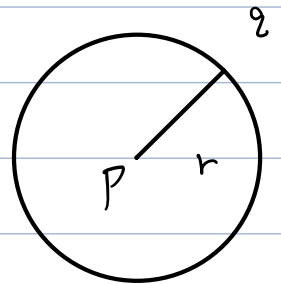
Corollary Let $B_R(p)$ be a geodesic ball $0 < r < R$, let $q \in S_r(p)$ Then

$$d(p, q) = r$$

Proof. $q = \exp_p(w)$, $|w| = r$

$$\gamma(t) = \exp_p(tw)$$

$$d(p, q) = L(\gamma) = |w| \cdot 1 = r$$



Corollary Let $B_R(p)$ be a geodesic ball

$$B_R(p) = \mathcal{B}_R(p) = \{x \in M : d(x, p) < R\}.$$

Proof.

$$\textcircled{1} B_R(p) \subset \mathcal{B}_R(p)$$

Let $q \in B_R(p)$, $q \neq p$, $q \in S_r(p)$.

$$d(p, q) = r < R.$$

$$\text{i.e. } q \in \mathcal{B}_R(p)$$

$$② \mathcal{B}_R(p) \subset \mathcal{B}_R(p)$$

It suffices to check that

$$\mathcal{B}_R(p)^c \subset \mathcal{B}_p(p)^c$$

i.e. if $q \notin \mathcal{B}_R(p)$, then $d(p, q) \geq R$.

Given.

$$\alpha: [a, b] \rightarrow M \quad \alpha(a) = p, \quad \alpha(b) = q.$$

By compactness, $\exists c \in [a, b]$ such that

$$\alpha(c) \in S_r(p). \quad (\text{for } 0 \leq r < R)$$

$$L(\alpha) = L(\alpha, [\alpha, c]) + L(\alpha, [c, b])$$

$$\geq d(p, \alpha(c)) = r$$

Let $r \rightarrow R^-$, we have

$$L(\alpha) \geq R.$$

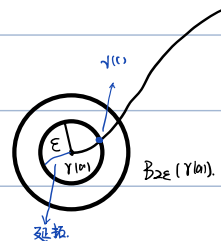
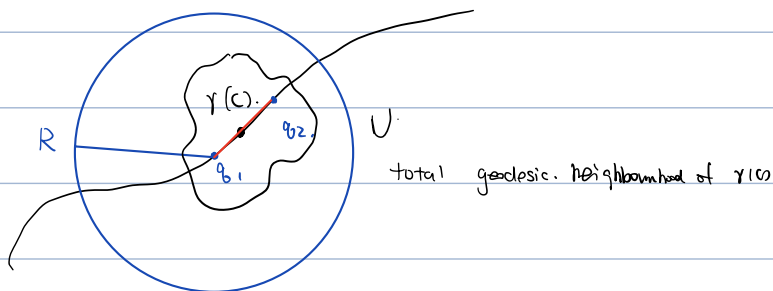
$$\Rightarrow d(p, q) = \inf L(\alpha) \geq R$$

Corollary. Let $\gamma: [a, b] \rightarrow M$ be piecewise smooth and $|\gamma'| = c, c \neq 0$.

If γ is minimizing, then γ is a geodesic segment.

~~Proof~~. $\gamma: [0, c] \rightarrow M$ minimizing (for all c)

Then it's a radial geodesic.



$\exists r > 0$ such that $\forall q \in U, U \subset B_r(q)$

$$\gamma((c-\epsilon, c+\epsilon)) \subset U$$

$$q_1 = \gamma(c-\epsilon) \quad q_2 = \gamma(c+\epsilon)$$

$$q_1 \rightsquigarrow q_2 \text{ geodesic} \Rightarrow \gamma''(c) = 0$$

Theorem. Let $p \in M$. Suppose $\text{Dom}(\exp_p) = T_p M$. Let $q \in M$.

$$d(p, q) = r > 0.$$

Then $\exists v \in T_p M$, $|v| = 1$ such that,

$$\exp_p(rv) = q.$$

Proof.

① Determine v .

$\exists 0 < \varepsilon < r$ $\exists B_{\varepsilon}(p)$ is a geodesic ball

$\exists m \in S_{\varepsilon}(p)$ such that.

$$d(m, q) = \min_{m \in S_{\varepsilon}(p)} d(m, q)$$

Let $m = \exp_p(\varepsilon v)$ $|v| = 1$ Denote

$$\gamma(t) = \exp_p(tv) \quad t \in \mathbb{R}.$$

It suffices to prove $\gamma(r) = q$.

Continuation. method.

Claim 1

$$L(\gamma; [0, \varepsilon]) + d(\gamma(\varepsilon), q) = d(p, q).$$

Proof. It suffices to prove that.

$$L(\gamma; [0, \varepsilon]) + d(\gamma(\varepsilon), q) \leq d(p, q).$$

Incl. $\forall \delta > 0$, $\exists \alpha : [0, 1] \rightarrow M$. $\alpha(0) = p$, $\alpha(1) = q$, $\alpha(\hat{t}) \in S_{\varepsilon}(p)$

$$\text{thus. } d(p, q) + \delta \geq L(\alpha; [0, 1]) = L(\alpha; [0, \hat{t}]) + L(\alpha; [\hat{t}, 1])$$

$$\geq L(\gamma; [0, \varepsilon]) + d(\gamma(\varepsilon), q)$$

Let $\delta \rightarrow 0^+$. $\square$.

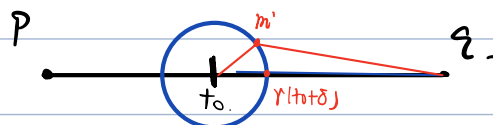
Def :

$$T = \{ t \in [0, r]; L(\gamma; [0, t]) + d(\gamma(t), q) = d(p, q) \}.$$

$$\Rightarrow \varepsilon \in T \Rightarrow T \neq \emptyset.$$

Claim 2 :

Assume $t_0 \in T$, $0 < t_0 < r$, $\exists 0 < \delta < r - t_0$ such that $t_0 + \delta \in T$



$$\left[\begin{array}{l} L(\gamma; [t_0, t_0]) + d(\gamma(t_0), q) \leq d(p, q) \\ L(\gamma; [0, t_0 + \delta]) + d(\gamma(t_0 + \delta), q) \leq d(p, q) \end{array} \right]$$

~~Proof~~. $\exists 0 < \delta < r - t_0$ such that $B_{2\delta}(\gamma(t_0))$ a geodesic. Let $m' \in S_\delta(\gamma(t_0))$ such that

$$d(m', q) = \min_{m \in S_\delta(\gamma(t_0))} d(m, q)$$

Let

$$m' = \exp_{\gamma(t_0)}(\delta w) \quad |w| = 1$$

$$\alpha(t) = \exp_{\gamma(t_0)}(tw) \quad 0 \leq t \leq \delta.$$

It suffices to show that

$$\gamma(t) = \alpha(t - t_0) \quad t_0 \leq t \leq t_0 + \delta$$

$$\left. \begin{array}{l} L(\gamma; [t_0, t_0]) + d(\gamma(t_0), q) = d(p, q) \\ L(\alpha; [0, \delta]) + d(\alpha(\delta), q) = d(\alpha(t_0), q) \end{array} \right\} \begin{array}{l} \text{if } \alpha(t - t_0) = \gamma(t) \\ \text{then. Claim obtained.} \end{array}$$

$p \rightsquigarrow m'$ 最短 $\Rightarrow p \rightsquigarrow m'$ 测地线.

Let $B_R(p)$ be a geodesic ball, $\gamma: [0, \varepsilon] \rightarrow M$ be a geodesic segment with $\gamma(0) = p$ and $|\gamma'| \equiv 1$. If

$$\gamma([0, \varepsilon]) \subset B_R(p)$$

then. $\varepsilon < R$.

~~Proof~~. $\exists 0 < r < R$ such that,

$$\gamma([0, \varepsilon]) \subset B_r(p).$$

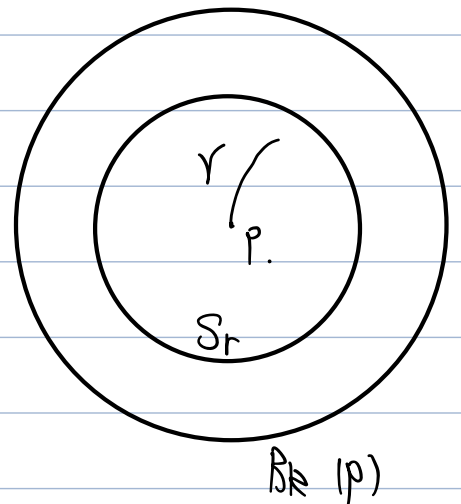
$$\text{Let } \alpha(t) = \exp_p(t \gamma'(0)) \quad 0 \leq t \leq r$$

Since $\alpha'(0) = \gamma'(0)$, we have.

$$\alpha(t) = \gamma(t) \quad \forall t \in [0, r]$$

Assume $\varepsilon > r$.

$$\gamma(r) = \alpha(r) \in S_r(p).$$



4.1

Theorem 1. Let $p \in M$. Assume that $\text{Dom}(\exp_p) = T_p M$. Let $q \in M$, $r = d(p, q) > 0$. Then $\exists v \in T_p M$, $|v| = 1$
 $\exp_p(rv) = q$.

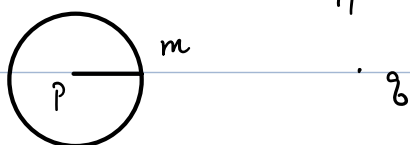
Proof.

Determine v .

$\exists 0 < \varepsilon < r$. such that: $B_{\varepsilon}(p)$ in a geodesic ball. Take $m \in S_{\varepsilon}(p)$, such that
 $d(m, q) = \min \{d(x, q) : x \in S_{\varepsilon}(p)\}$.

Assume $m = \exp_p(\varepsilon v)$, $v \in T_p M$, $|v| = 1$. Denote

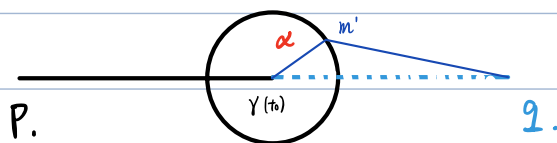
$$\gamma(t) = \exp_p(tv)$$



· Claim 1: $L(\gamma; [0, \varepsilon]) + d(\gamma(\varepsilon), q) = d(p, q)$

Denote: $T = \{t \in [0, r] : L(\gamma; [0, t]) + d(\gamma(t), q) = d(p, q)\}$

· Claim 2: Suppose $t_0 \in T$, $0 < t_0 < r$, then, $\exists 0 < \delta < r - t_0$ such that $t_0 + \delta \in T$.



Given $t_0 \in (0, r)$, $\exists \delta \in (0, d(\gamma(t_0), q))$ such that. $B_{\delta}(\gamma(t_0))$ is a geodesic ball, $\exists m' \in S_{\delta}(\gamma(t_0), \delta)$ such that

$$d(m', q) = \min \{d(x, q) : x \in S_{\delta}(\gamma(t_0))\}$$

Let $m' = \exp_{\gamma(t_0)}(\delta w)$. Denote:

$$\alpha(t) = \exp_{\gamma(t_0)}(tw) \quad t \in [0, \delta].$$

Denote.

$$\tilde{\gamma} = \gamma|_{[0, t_0]} + \alpha.$$

Since $t_0 \in T$, we have

$$\begin{cases} L(\gamma; [0, t_0]) + d(\gamma(t_0), q) = d(p, q). \\ L(\alpha; [0, \delta]) + d(\alpha(\delta), q) = d(\alpha(0), q) \end{cases}$$

$$\Rightarrow L(\gamma; [0, t_0]) + L(\alpha; [t_0, \delta]) + d(\alpha(\delta), q) = d(p, q)$$

$$\Rightarrow L(\tilde{\gamma}; [0, t_0 + \delta]) + d(\tilde{\gamma}(t_0 + \delta), q) = d(p, q)$$

$$\Rightarrow L(\tilde{\gamma}; [0, t_0 + \delta]) = d(p, q) - d(\tilde{\gamma}(t_0 + \delta), q) \leq d(p, \tilde{\gamma}(t_0 + \delta))$$

$$\Rightarrow L(\tilde{\gamma}; [0, t_0 + \delta]) = d(p, \tilde{\gamma}(t_0 + \delta))$$

$\Rightarrow \tilde{\gamma}$ is minimizing

Since $|\tilde{\gamma}'| \equiv 1$, $\tilde{\gamma}$ is a geodesic. But $\tilde{\gamma}'(0) = \gamma'(0)$, we have
 $\tilde{\gamma} = \gamma|_{[0, t_0 + \delta]}$.

It follows that.

$$L(\gamma; [0, t_0 + \delta]) + d(\gamma(t_0 + \delta), q) = d(p, q)$$

i.e. $t_0 + \delta \in T$.

• Claim 3: $r \in T$

By Claim 2: $\sup T = r$. Then, $\exists \{t_k\} \subset T$, $t_k \rightarrow r$.

$$\text{i.e. } L(\gamma; [0, t_k]) + d(\gamma(t_k), q) = d(p, q)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$L(\gamma; [0, r]) + d(\gamma(r), q) = d(p, q)$$

□

Theorem (Hopf - Rinow)

Let $p \in M$, the following statement is equivalence.

① $\text{Dom}(\exp_p) = T_p M$ (given p).

②. each bounded closed subset of M is compact.

③ M is a complete metric space.

④. M is geodesically complete $\leadsto \forall q \in M$, $\text{Dom}(\exp_q) = T_q M$. (for all q).

Moreover, each of ① $\sim$ ④ implies

⑤. $\forall q_1, q_2 \in M$, $q_1 \neq q_2$, $\exists$ a minimizing geodesic joining q_1 to q_2 .

Proof.

④ $\Rightarrow$ ⑤:

$\text{Dom}(\exp_{q_1}) = T_{q_1} M$ and theorem 1.

① $\Rightarrow$ ②

Let $A \subset M$ bounded and closed. $\text{Dom}(\exp_p) = T_p M$.

$$A \subset \overline{B_R(p)} = \{x \in M : d(x, p) \leq R\}.$$

Let $q \in \overline{B_R(p)}$, $d(p, q) = r < R$. $\exists v \in T_p M$, $|v| = 1$

$$\exp_p(rv) = q.$$

Let $w = rv$, $|w| = r$, $\exp_p(w) = q$

$$\Rightarrow q \in \exp_p(\overline{B_R(0_p)})$$

$$\Rightarrow \overline{B_R(p)} \subset \exp_p(\underbrace{\overline{B_R(0_p)}}_{\subset \mathbb{R}^n \text{ compact}}).$$

$\overline{B_R(p)}$ compact.

② $\Rightarrow$ ③ obvious.

③ $\Rightarrow$ ④ :

Let $\gamma : (a, b) \rightarrow M$ solve.

$$\begin{cases} \gamma'' = 0 \\ \gamma'(0) = v \end{cases} \quad |v| = 1$$

It suffices to check that $a = -\infty$, $b = +\infty$.



Suppose, by contradiction, that $b < +\infty$. $\forall \{t_i\} \subset (a, b)$, $t_i \rightarrow b$

$$d(\gamma(t_i), \gamma(t_j)) \leq |t_i - t_j|$$

$\Rightarrow \{\gamma(t_i)\}$ Cauchy, It follows that $\exists m \in M$, $\gamma(t_i) \rightarrow m$ (M complete).

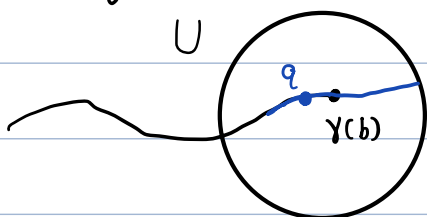
Thus, $\gamma(t) \rightarrow m$ as $t \rightarrow b^-$

Define: $\gamma(b) := \lim_{t \rightarrow b^-} \gamma(t)$.

Then $\gamma : (a, b] \rightarrow M$. continuous, $\rightarrow$ but not smooth. consider $\gamma(b-\epsilon)$. Smooth. $\uparrow$

Let U be a totally geodesic neighbourhood of $\gamma(b)$. Thus $\exists R > 0$, such that. $\forall q \in U$.

$B_R(q)$ is a geodesic ball, $B_R(q) \supset U$.



Then, $\exists 0 < \epsilon < \min(\frac{b-a}{2}, r)$ such that.

$$\gamma([b-\epsilon, b]) \subset U$$

$$\text{Let } \gamma = \gamma(b-\varepsilon). \quad \omega = \gamma'(b-\varepsilon)$$

α extend γ from $(a, b] \rightsquigarrow (a, b-\varepsilon+r)$

$b < b-\varepsilon+r$ contradiction.

④ $\Rightarrow$ ① obvious.

4.3

Def. M is a Riemann manifold. Define

$$R: \mathcal{V}^3(M) \rightarrow \mathcal{V}(M)$$

$$(X, Y, Z) \mapsto D_Y D_X Z - D_X D_Y Z - D[X, Y] Z$$

R is a 4-tensor. We call R the curvature tensor of M .

(3,1)

R is $\mathcal{F}$ -linear for each argument.

$$R(fX, Y, Z) = f R(X, Y, Z)$$

$$R(X, fY, Z) = f R(X, Y, Z)$$

$$R(X, Y, fZ) = f R(X, Y, Z) \quad fX Y - Y fX$$

$$\begin{aligned} R(fX, Y, Z) &= D_Y D_{fX} Z - D_{fX} D_Y Z - D[Y, fX] Z \\ &= D_Y (f D_X Z) - f D_X D_Y Z - D(Y(fX) - fX(Y)) Z \\ &= f D_Y D_X Z + Y f D_X Z - f D_X D_Y Z \\ &\quad - D(Y f X - f X Y) Z \\ &= f (D_Y D_X Z - D_X D_Y Z) + Y f D_X Z - D(Y f X - f X Y) Z \\ &= f (D_Y D_X Z - D_X D_Y Z - D[Y, X] Z) \\ &= f R(X, Y, Z) \end{aligned}$$

$$\begin{aligned} R(X, Y, fZ) &= D_Y D_X (fZ) - D_X D_Y (fZ) - D[X, Y] fZ \\ &= D_Y (X f Z + f D_X Z) - D_X (Y f Z + f D_Y Z) \\ &\quad + (X Y f - Y X f) Z + f D[X, Y] Z \\ &= Y X f Z + \underline{X f D_Y Z} + \underline{Y f D_X Z} + f D_Y D_X Z \\ &\quad - (X Y f) Z + \underline{Y f D_X Z} + \underline{X f D_Y Z} + f D_X D_Y Z \end{aligned}$$

$$\begin{aligned}
& + [X, Y]fZ + fD[X, Y]Z \\
& = fD_Y D_X Z - fD_X D_Y Z - fD[Y, X]Z \\
& = f(D_Y D_X Z - D_X D_Y Z - D[Y, X]Z) \\
& = fR(X, Y, Z)
\end{aligned}$$

$$M = \mathbb{R}^n, \quad Z = z^i e_i$$

$$\begin{cases} D_Y D_X Z = D_Y X(z^i) e_i = YX(z^i) e_i \\ D_X D_Y Z = XY(z^i) e_i \end{cases}$$

$$\hookrightarrow D_Y D_X Z - D_X D_Y Z = D[Y, X]Z$$

$$R(X, Y, Z, W) = \langle R(X, Y, Z), W \rangle$$

$$R(X, Y) = D_Y D_X - D_X D_Y - D[Y, X] : \mathcal{V}(M) \longrightarrow \mathcal{V}(M)$$

curvature operation

$$R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = D_{\frac{\partial}{\partial x^j}} D_{\frac{\partial}{\partial x^i}} - D_{\frac{\partial}{\partial x^i}} D_{\frac{\partial}{\partial x^j}}$$

Theorem

- ① $R(X, Y, Z, W) = -R(Y, X, Z, W)$
- ② $R(X, Y, Z, W) = -R(X, Y, W, Z)$
- ③ $R(X, Y, Z, W) + R(Z, X, Y, W) + R(Y, Z, X, W) = 0$
- ④ $R(X, Y, Z, W) = R(Z, W, X, Y)$

Proof: ① obvious.

$$\begin{aligned}
\textcircled{2}: \quad R(X, Y, Z, Z) &= \langle D_Y D_X Z, Z \rangle - \langle D_X D_Y Z, Z \rangle - \langle D[Y, X]Z, Z \rangle \\
&= \frac{1}{2} YX \langle Z, Z \rangle - \langle D_X Z, D_Y Z \rangle \\
&= \frac{1}{2} XY \langle Z, Z \rangle - \langle D_X Z, D_Y Z \rangle \\
&= \langle D_Y D_X Z, Z \rangle - \langle D_X D_Y Z, Z \rangle = \frac{1}{2} (YX - XY) \langle Z, Z \rangle
\end{aligned}$$

$$= \frac{1}{2} [Y, X] \langle Z, Z \rangle = \langle D[X, Y] Z, Z \rangle$$

$$\textcircled{3}. R(X, Y, Z) = \underline{D_Y D_X Z} - \underline{D_X D_Y Z} - D[X, Y] Z$$

$$R(Z, X, Y) = \underline{D_X D_Z Y} - \underline{D_Z D_X Y} - D[Z, X] Y$$

$$R(Y, Z, X) = \underline{D_Z D_Y X} - \underline{D_Y D_Z X} - D[Z, Y] X$$

$$\Rightarrow R(X, Y, Z) + R(Z, X, Y) + R(Y, Z, X)$$

$$= D_Y [X, Z] + D_X [Z, Y] + D_Z [Y, X]$$

$$- D[X, Z] Y - D[Z, Y] X - D[Y, X] Z$$

$$= [Y, [X, Z]] + [X, [Z, Y]] + [Z, [Y, X]] = 0.$$

$$\textcircled{4} \quad \underline{R(X, Y, Z, W)} = -R(Z, X, Y, W) - R(Y, Z, X, W)$$

$$= R(Z, X, W, Y) + R(Y, Z, W, X)$$

$$= -\underline{R(W, Z, X, Y)} - R(X, W, Z, Y)$$

$$- R(W, Y, Z, X) - \underline{R(Z, W, Y, X)}$$

$$= 2R(Z, W, X, Y) + R(X, W, Y, Z) + R(W, Y, X, Z)$$

$$= 2R(Z, W, X, Y) - R(Y, X, W, Z)$$

$$= 2R(Z, W, X, Y) - \underline{R(X, Y, Z, W)}.$$

Theorem 2

$$D_X R(Y, Z, W) + D_Z R(X, Y, W) + D_Y R(Z, X, W) = 0.$$

$$(d R(Y, Z, W, X) = 0)$$

$$\text{Where: } D_X R(Y, Z, W) = D_X (R(Y, Z, W)) - R(D_X Y, Z, W) \\ - R(Y, D_X Z, W) - R(Y, Z, D_X W)$$

Notion:

L : vector space, A, B, C : linear operator on L .

$$[A, B] = AB - BA$$

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

$$F: \mathcal{V}^4(\mathcal{M}) \rightarrow \mathcal{V}(\mathcal{M}), F(X, Y, Z, W)$$

$$\sigma F(X, Y, Z, W) := F(X, Y, Z, W) + F(Y, Z, X, W) + F(Z, X, Y, W)$$

Proof of theorem 2.

$$\begin{aligned} D_x R(Y, Z, W) &= D_x (R(Y, Z)W) - R(D_x Y, Z)W \\ &\quad - R(Y, D_x Z)W - R(Y, Z)D_x W \\ &= [D_x, R(Y, Z)]W \\ &\quad - R(Y, D_x Z)W - R(Y, Z)D_x W \end{aligned}$$

$\Downarrow$

$$\begin{aligned} \sigma D_x(Y, Z, W) &= \sigma [D_x, R(Y, Z)]W \\ &\quad - \sigma R(Y, D_x Z)W \\ &\quad - \sigma R(Y, Z)D_x W. \end{aligned}$$

$$R(Y, Z) = D_Z D_Y - D_Y D_Z - D[Z, Y] = [D_Z, D_Y] - D[Z, Y]$$

$\Downarrow$

$$\begin{aligned} &\sigma [D_x, R(Y, Z)]W \\ &= \sigma [D_x, [D_Z, D_Y]]W - \sigma [D_x, D[Z, Y]]W \\ &= -\sigma [D_x, D[Z, Y]]W \end{aligned}$$

$$\begin{aligned} &\sigma R(D_x Y, Z)W + \sigma R(Y, D_x Z)W \\ &= \sigma R(D_Z X, Y)W + \sigma R(Y, D_x Z)W \\ &= \sigma R(D_Z X, Y)W - \sigma R(D_x Z, Y)W \\ &= \sigma R([Z, X], Y)W. \\ &= \sigma [D_Y, D[Z, X]]W - \sigma D[Y, [Z, X]]W \\ &= \sigma [D_Y, D[Z, X]]W. \end{aligned}$$

$$\begin{aligned} \sigma D_x R(Y, Z, W) &= -\sigma [D_x, D[Z, Y]]W - \sigma [D_Y, D[Z, X]]W \\ &= -\sigma [D_Y, D[X, Z]]W - \sigma [D_Y, D[Z, X]]W \\ &= \sigma [D_Y, D[Z, X]]W - \sigma [D_Y, D[Z, X]]W = 0. \end{aligned}$$

$$\textcircled{1}: R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = R_{ijk}^l \frac{\partial}{\partial x^l}$$

$$R_{ijk}^l = (\frac{\partial}{\partial x^j} \Gamma_{ik}^l - \frac{\partial}{\partial x^i} \Gamma_{jk}^l + \Gamma_{ik}^m \Gamma_{jm}^l - \Gamma_{jk}^m \Gamma_{im}^l)$$

Proof. $R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = D_{\frac{\partial}{\partial x^j}} D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} - D_{\frac{\partial}{\partial x^i}} D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}$

$$D_{\frac{\partial}{\partial x^i}} D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = D_{\frac{\partial}{\partial x^j}} (\Gamma_{ik}^m \frac{\partial}{\partial x^m})$$

$$= \frac{\partial}{\partial x^j} \Gamma_{ik}^m \frac{\partial}{\partial x^m} + \Gamma_{ik}^m D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^m}$$

$$= (\frac{\partial}{\partial x^j} \Gamma_{ik}^l + \Gamma_{ik}^m \Gamma_{jm}^l) \frac{\partial}{\partial x^l}$$

$$D_{\frac{\partial}{\partial x^j}} D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} = (\frac{\partial}{\partial x^i} \Gamma_{jk}^l + \Gamma_{jk}^m \Gamma_{im}^l) \frac{\partial}{\partial x^l}$$

$$R_{ijk}^l = (\frac{\partial}{\partial x^j} \Gamma_{ik}^l - \frac{\partial}{\partial x^i} \Gamma_{jk}^l + \Gamma_{ik}^m \Gamma_{jm}^l - \Gamma_{jk}^m \Gamma_{im}^l)$$

$$\textcircled{2}. R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^m}) = R_{ijkl} = R_{ijk}^m g_{ml}$$

$$\text{or } R_{ijk}^m = R_{ijkl} g^{lm}$$

By symmetry:

$$\begin{cases} R_{ijkl} = -R_{jikl} \\ R_{ijkl} = -R_{ijlk} \\ R_{ijkl} + R_{kijl} + R_{jkil} = 0 \end{cases} \quad \frac{1}{12} n^2 (n+1)$$

Prop. Let X be a smooth vector field along a smooth surface f

Denote $f = f(s, t)$ $X = X(s, t)$ Then.

$$\frac{\partial^2 X}{\partial s \partial t} - \frac{\partial^2 X}{\partial t \partial s} = R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}) X$$

Proof. $\forall p \in M$, let $(U, \phi) = (U, x^1, \dots, x^n)$ be a local chart near p , and.

$$X = X^i(s, t) \frac{\partial}{\partial x^i} |_{f(s, t)}$$

$$\frac{\partial}{\partial t} X = \frac{\partial}{\partial t} X^i(s, t) \frac{\partial}{\partial x^i} + X^i(s, t) D_{\frac{\partial f}{\partial t}} \frac{\partial}{\partial x^i}$$

$$= \partial_t X^i \partial_i + X^i \partial_t f^j D_{\partial_j} \partial_i$$

$$\frac{\partial}{\partial s} \frac{\partial}{\partial t} X = \partial_s \partial_t X^i \frac{\partial}{\partial x^i} + \partial_t X^i \partial_s f^j D_{\partial_j} \partial_i + \partial_s X^i \partial_t f^j D_{\partial_j} \partial_i \\ + X^i \partial_s \partial_t f^j D_{\partial_j} \partial_i + X^i \partial_t f^j \partial_s f^k D_{\partial_k} D_{\partial_j} \partial_i$$

$$\frac{\partial}{\partial t} \frac{\partial}{\partial s} X = \partial_t \partial_s X^i \frac{\partial}{\partial x^i} + \partial_s X^i \partial_t f^j D_{\partial_j} \partial_i + \partial_t X^i \partial_s f^j D_{\partial_j} \partial_i$$

$$\begin{aligned}
 & + X^i \partial_s \partial_t f^j D_{\partial_j} \partial_i + X^i \partial_s f^j \partial_t f^k D_{\partial_k} D_{\partial_j} \partial_i \\
 \frac{\partial}{\partial s} \frac{\partial}{\partial t} X - \frac{\partial}{\partial t} \frac{\partial}{\partial s} X &= X^i \partial_t f^j \partial_s f^k (D_{\partial_k} D_{\partial_j} \partial_i - D_{\partial_j} D_{\partial_k} \partial_i) \\
 &= X^i \partial_t f^j \partial_s f^k R(\partial_j, \partial_k) \partial_i \\
 &= R(\partial_t f, \partial_s f) X
 \end{aligned}$$

4.8

Sectional curvature.

Def 1. Define.

$$\text{Sec} : \bigcup_{p \in M} (T_p M)^2 \rightarrow \mathbb{R}$$

$$(u, v) \mapsto R(u, v, u, v)$$

$$\textcircled{1} \text{Sec}(u, u) = 0.$$

$$\textcircled{2} \text{Sec}(u, v) = \text{Sec}(v, u)$$

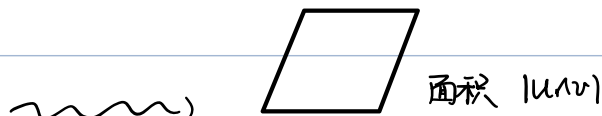
$$\textcircled{3} \text{Sec}(s u + t v) = s^2 t^2 \text{Sec}(u, v) \quad s, t \in \mathbb{R}$$

$$\dim M = 1 \Rightarrow R(u, v, x, y) = 0 \quad \text{Hereafter: we assume } \dim M \geq 2$$

$$u, v \in T_p M, \quad u \neq v.$$

$$u \wedge v =: \text{span}\{u, v\} : 2\text{-plane}$$

$$|u \wedge v| = \sqrt{|u|^2 |v|^2 - \langle u, v \rangle^2}$$



Def. Let

$$\Sigma = \{ \sigma : \sigma \text{ is a 2-plane in } T_p M, p \in M \}$$

Define

$$\text{Sec} : \Sigma \rightarrow \mathbb{R}.$$

$$u \wedge v \mapsto \frac{\text{Sec}(u, v)}{|u \wedge v|^2} \quad (*).$$

Remark: If M : regular surface in $\mathbb{R}^3$ $\dim T_p M = 2. \Rightarrow \Sigma = \{T_p M\}.$

In this case. $\text{Sec}(T_p M) = \text{Gauss curvature}$

Remark: (*) is well defined. Let

$$v = u \wedge v = e_1 \wedge e_2$$

where $\langle e_i, e_j \rangle = \delta_{ij}.$

$$u = u^i e_i \quad v = v^i e_i$$

$$\begin{aligned}
\Rightarrow. \quad \text{Sec}(u, v) &= R(u, v, u, v) \\
&= u^i v^j R(e_i, e_j, u, v) \\
&= u^1 v^2 R(e_1, e_2, u, v) + u^2 v^1 R(e_1, e_2, u, v) \\
&= (u^1 v^2 - u^2 v^1) R(e_1, e_2, u, v) \\
&= (u^1 v^2 - u^2 v^1)^2 R(e_1, e_2, e_1, e_2)
\end{aligned}$$

$$|u \wedge v| = |u^1 v^2 - u^2 v^1|$$

$$\Rightarrow. \quad \text{Sec}(e_1 \wedge e_2) = \text{Sec}(u \wedge v)$$

Theorem. Let $u, v, x, y \in T_p M$, Then

$$R(x, y, u, v) = \frac{1}{6} \frac{\partial^2}{\partial s \partial t} \Big|_{(0,0)} (\text{Sec}(x+su, y+tv) - \text{Sec}(x+sv, y+tu))$$

Proof. $\text{Sec}(x+su, y+tv) = R(x+su, y+tv, x+su, y+tv)$

$$\begin{aligned}
&= s^2 R(u, y+tv, u, y+tv) + s R(u, y+tv, x, y+tv) \\
&\quad + s R(x, y+tv, u, y+tv) + R(x, y+tv, x, y+tv)
\end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial s} \Big|_0 \text{Sec}(x+su, y+tv) = 2R(x, y+tv, u, y+tv)$$

$$\Rightarrow \frac{\partial}{\partial t} \Big|_0 R(x, y+tv, u, y+tv) = R(x, v, u, y) + R(x, y, u, v)$$

It follows that

$$\begin{aligned}
&\frac{\partial^2}{\partial t \partial s} \Big|_{(0,0)} (\text{Sec}(x+su, y+tv) - \text{Sec}(x+sv, y+tu)) \\
&= 2(R(x, v, u, y) + R(x, y, u, v) - R(x, u, v, y) - R(x, y, v, u))
\end{aligned}$$

But

$$\begin{aligned}
&R(x, v, u, y) - R(x, u, v, y) \\
&= R(u, y, x, v) - R(u, x, y, v) \\
&= -R(y, x, u, v) - R(x, u, y, v) - R(u, x, y, v) \\
&= R(x, y, u, v)
\end{aligned}$$

Def. If

$$\text{Sec}(\sigma) = k \quad \forall \sigma \in \Sigma$$

We say M is a manifold with constant sectional curvature k .

Example. $\begin{cases} \mathbb{R}^n \rightsquigarrow 0 \\ S^n \rightsquigarrow 1 \\ H^n \rightsquigarrow -1 \end{cases}$ complete simply connected.

Theorem. M has constant sectional curvature k iff

$$R(x, y, u, v) = k(\langle x, u \rangle, \langle y, v \rangle - \langle x, v \rangle \cdot \langle y, u \rangle)$$

Proof .

$$\Leftarrow \forall \sigma \in \Sigma \quad \sigma = p_1 \lambda p_2 \quad , \quad p_i e_j = \delta_{ij}.$$

$$\text{Sec}(e_1, e_2) = R[e_1, e_2, p_1, p_2] = k$$

“ $\Rightarrow$ ”

$$R(x, y, u, v) = \frac{1}{6} \frac{\partial^2}{\partial s \partial t} \Big|_{(0,0)} \left(\sec(x+su, y+tv) - \sec(x+sv, y+tu) \right) \\ = \frac{k}{6} \frac{\partial^2}{\partial s \partial t} \Big|_{(0,0)} \left(|(x+su) \wedge (y+tv)|^2 - |(x+sv) \wedge (y+tu)|^2 \right) \\ = \frac{k}{6} \frac{\partial^2}{\partial s \partial t} \Big|_{(0,0)} \left(|(x+su) \wedge (y+tv)|^2 - |(x+sv) \wedge (y+tu)|^2 \right) \\ = \frac{k}{6} (I - II)$$

$$= |x+su|^2 |y+tv|^2 - \langle x+su, y+tv \rangle^2$$

$$I = 4 \langle x, u \rangle \langle y, v \rangle - 2 (\langle x, v \rangle \langle u, y \rangle + \langle x, y \rangle \langle u, v \rangle)$$

$$\text{II} = 4 \langle x, v \rangle \langle y, u \rangle - 2(\langle x, u \rangle \langle y, v \rangle + \langle \quad \rangle \langle \quad \rangle)$$

$$\Rightarrow \text{I} - \text{II} = 6 (\langle x, u \rangle \langle y, v \rangle - \langle x, v \rangle \langle u, y \rangle)$$

$$\Rightarrow R(x, y, u, v) = k (\langle x, u \rangle \langle y, v \rangle - \langle x, v \rangle \langle y, u \rangle)$$

Let $f(z) = \frac{az+b}{cz+d}$, $z = x+iy$, $ad-bc=1$. $a, b, c, d \in \mathbb{R}$.

①. $f(H^2) = H^2$

$$\forall z \in \mathbb{H}^2, \operatorname{Im} z > 0, \text{ thus.} \quad \alpha c |z|^2$$

$$\operatorname{Im} f(z) = |cz+d|^{-1} \operatorname{Im}(az+b)(c\bar{z}+d)$$

$$\begin{aligned}
&= |cz+d|^{-1} \operatorname{Im} (a|z|^2 + bc\bar{z} + adz + bd) \\
&= |cz+d|^{-1} \cdot (ad-bc) \operatorname{Im} z = |cz+d|^{-1} \operatorname{Im} z > 0
\end{aligned}$$

thus $f(z) \in \mathbb{H}^2$.

② $f(\mathbb{R}i) =$ ~~垂直~~ x 轴由直线或圆 圆心在 x 轴上圆.

Let $w = f(z) = \frac{az+b}{cz+d}$, then

$$\begin{aligned}
z &= \frac{dw-b}{a-cw} = |a-cw|^{-1} (dw-b)(a-c\bar{w}) \\
&= |a-cw|^{-1} (adw + bc\bar{w} - cd|w|^2 - ba)
\end{aligned}$$

But $z \in \mathbb{R}i$ implies that $\operatorname{Re} z = 0$.

$$\text{i.e. } -cd|w|^2 + (ad+bc) \operatorname{Re} w - ba = 0.$$

③. $f: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is Riemann isometry.

Let $z(t) = x(t) + iy(t)$ be a curve in $\mathbb{H}^2$,

$$\begin{aligned}
df(z') &= df\left(\frac{d}{dt} z\right) = \frac{d}{dt} f \circ z \\
&= f'(z(t)) z'(t) = \frac{z'}{(cz+d)^2}
\end{aligned}$$

It follows that

$$\|df(z')\| = \frac{|z'|}{|cz+d|^2} / |\operatorname{Im} f \circ z| = \frac{|z'|}{|cz+d|^2} \cdot \frac{|cz+d|^2}{y} = \frac{|z'|}{y} = \|z'\|.$$

4.101

$$H^n = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n : x^n > 0\}$$

$$g_{ij}(x) = \frac{\delta_{ij}}{(x^n)^2}$$

$$F = x^n \quad f = \log F$$

$$\text{notation: } g_{ij} = F^{-2} \delta_{ij} \quad g^{ij} = F^2 \delta^{ij}$$

$$V_i = \frac{\partial V}{\partial x^i} \quad V_{ij} = \frac{\partial^2 V}{\partial x^i \partial x^j} \quad F_i = F f_i$$

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)$$

$$\begin{aligned} \Gamma_{ij}{}^k &= \sum_m \Gamma_{ijm} g^{mk} = F^2 \sum_m \Gamma_{ijm} \delta^{mk} = F^2 \Gamma_{ijk} \\ &= \frac{1}{2} F^2 \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \end{aligned}$$

if i, j, k are distinct, $\Gamma_{ij}{}^k = 0$. Assume $i \neq j$.

$$\textcircled{1} \Gamma_{ij}{}^i = \frac{1}{2} F^2 \cdot \frac{\partial g_{ii}}{\partial x^j} = \frac{1}{2} F^2 \frac{\partial}{\partial x^j} \frac{1}{F^2} = \frac{1}{2} F^2 \cdot -2 \frac{1}{F^3} \cdot F f_j = -f_j$$

$$\textcircled{2} \Gamma_{ij}{}^j = \frac{1}{2} F^2 \cdot \frac{\partial g_{jj}}{\partial x^i} = \frac{1}{2} F^2 \frac{\partial}{\partial x^i} \frac{1}{F^2} = -f_i$$

$$\textcircled{3} \Gamma_{ii}{}^i = -\frac{1}{2} F^2 \cdot \frac{\partial g_{ii}}{\partial x^i} = f_i$$

$$\textcircled{4} \Gamma_{ii}{}^i = \frac{1}{2} F^2 \left(\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i} \right) = -f_i$$

$$\Rightarrow \Gamma_{ij}{}^i = \Gamma_{ji}{}^i = \Gamma_{ii}{}^i = -\Gamma_{jj}{}^i = -f_i$$

$$\begin{aligned} R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) &= D_{\frac{\partial}{\partial x^j}} D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} - D_{\frac{\partial}{\partial x^i}} D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \\ &= D_{\frac{\partial}{\partial x^j}} \left(\Gamma_{ik}{}^m \frac{\partial}{\partial x^m} \right) - D_{\frac{\partial}{\partial x^i}} \left(\Gamma_{jk}{}^m \frac{\partial}{\partial x^m} \right) \\ &= \frac{\partial \Gamma_{ik}{}^m}{\partial x^j} \frac{\partial}{\partial x^m} + \Gamma_{ik}{}^m \Gamma_{jm}{}^l \frac{\partial}{\partial x^l} - \left(\frac{\partial \Gamma_{jk}{}^m}{\partial x^i} \frac{\partial}{\partial x^m} + \Gamma_{jk}{}^m \Gamma_{im}{}^l \frac{\partial}{\partial x^l} \right) \\ &= \left(\frac{\partial \Gamma_{ik}{}^l}{\partial x^j} + \Gamma_{ik}{}^m \Gamma_{jm}{}^l \right) \frac{\partial}{\partial x^l} - \left(\frac{\partial \Gamma_{jk}{}^l}{\partial x^i} + \Gamma_{jk}{}^m \Gamma_{im}{}^l \right) \frac{\partial}{\partial x^l} \end{aligned}$$

$$\Rightarrow R_{ijk}{}^l = \frac{\partial \Gamma_{ik}{}^l}{\partial x^j} - \frac{\partial \Gamma_{jk}{}^l}{\partial x^i} + \Gamma_{ik}{}^m \Gamma_{jm}{}^l - \Gamma_{jk}{}^m \Gamma_{im}{}^l$$

$$R_{ijkl} = \sum_s R_{ijk}{}^s g_{sl} = F^{-2} R_{ijkl}$$

$$= F^{-2} \left(\frac{\partial \Gamma_{ik}{}^l}{\partial x^j} - \frac{\partial \Gamma_{jk}{}^l}{\partial x^i} + \sum_m (\Gamma_{ik}{}^m \Gamma_{jm}{}^l - \Gamma_{jk}{}^m \Gamma_{im}{}^l) \right)$$

If i, j, k, l distinct $R_{ijkl} = 0$

$$\textcircled{1} R_{iikl} = 0$$

$$\begin{aligned}
 \textcircled{2} R_{ijle} &= F^{-2} \left(\frac{\partial \Gamma_{ii}^e}{\partial x^j} + \sum_m (\Gamma_{ii}^m \Gamma_{jm}^e - \Gamma_{ji}^m \Gamma_{im}^e) \right) \\
 &= F^{-2} \left(\frac{\partial f_e}{\partial x^j} + \Gamma_{ii}^j \Gamma_{jj}^e - \Gamma_{ji}^j \Gamma_{jj}^e + \Gamma_{ii}^e \Gamma_{je}^e - \Gamma_{ji}^e \Gamma_{ie}^e \right. \\
 &\quad \left. + \Gamma_{ii}^i \Gamma_{ji}^e - \Gamma_{ji}^i \Gamma_{ii}^e \right) \\
 &= F^{-2} \left(\frac{\partial f_e}{\partial x^j} + f_j f_e - 0 + f_e \cdot (-f_j) - 0 + 0 - (-f_j) f_e \right) \\
 &= F^{-2} (f_{je} + f_j f_e)
 \end{aligned}$$

$$\textcircled{3} R_{ijki} = -R_{jik} = -F^{-2} (f_{jk} + f_j f_k)$$

$$\textcircled{4} R_{ijje} = -R_{ijje} = -F^{-2} (-f_{ie} + f_i f_e)$$

$$\textcircled{5} R_{ijkj} = R_{ijk} = -F^{-2} (f_{ik} + f_i f_k)$$

$$\textcircled{6} R_{ijkk} = 0.$$

$$\textcircled{7} R_{iiij} = 0$$

$$\begin{aligned}
 \textcircled{8} R_{ijij} &= F^{-2} \left(\frac{\partial \Gamma_{ii}^j}{\partial x^j} - \frac{\partial \Gamma_{ji}^j}{\partial x^i} + \sum_m (\Gamma_{ii}^m \Gamma_{jm}^j - \Gamma_{ji}^m \Gamma_{im}^j) \right) \\
 &= F^{-2} (f_{jj} + f_{ii} - \sum_{m \neq ij} f_m^2 - \Gamma_{ji}^i \Gamma_{ii}^j - \Gamma_{ji}^j \Gamma_{ij}^j + \Gamma_{ii}^i \Gamma_{ji}^j + \Gamma_{ii}^j \Gamma_{jj}^j) \\
 &= F^{-2} (f_{jj} + f_{ii} - \sum_{m \neq ij} f_m^2 - (-f_j) \cdot f_j - (-f_i) (-f_i) + (-f_i) (-f_i) + (f_j) (-f_j)) \\
 &= F^{-2} (f_{ii} + f_{jj} - \sum_{m \neq ij} f_m^2 + f_j^2 - f_i^2) \\
 &= F^{-2} (f_{ii} + f_{jj} - \sum_{m \neq ij} f_m^2)
 \end{aligned}$$

$$\textcircled{9} R_{ijji} = -R_{ijij} = -F^{-2} (f_{ii} + f_{jj} - \sum_{m \neq ij} f_m^2)$$

$$\textcircled{10} R_{iiii} = R_{ijjj} = 0.$$

$$F = x^n, \quad f = \ln x^n$$

$$f_i = \delta_{in} \frac{1}{x^n} \quad f_{ij} = \delta_{in} \delta_{jn} \frac{-1}{(x^n)^2}$$

$$R_{ijie} = F^{-2} (f_{je} + f_j f_e) = F^{-2} (\delta_{jn} \delta_{en} \frac{-1}{(x^n)^2} + \delta_{jn} \delta_{en} \frac{1}{(x^n)^2}) = 0$$

$$R_{ijki} = F^{-2} (f_{jk} + f_j f_k) = 0$$

⋮

$$R_{ijij} = F^{-2} \left(\delta_{in} \frac{-1}{(x^n)^2} + \delta_{jn} \frac{-1}{(x^n)^2} - \sum_{m \neq ij} \delta_{mn} \frac{1}{(x^n)^2} \right)$$

$$i=n \text{ or } j=n \rightsquigarrow \frac{1}{(x^n)^2} \cdot \frac{-1}{(x^n)^2} = \frac{-1}{(x^n)^4}$$

$$i \neq n, j \neq n \rightsquigarrow \frac{1}{(x^n)^2} \cdot \frac{-1}{(x^n)^2} = \frac{-1}{(x^n)^4}$$

$$R_{ijkl} = -F^{-4} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$$

$$e_i = \frac{\partial}{\partial x^i} / \left| \frac{\partial}{\partial x^i} \right| = F \frac{\partial}{\partial x^i}$$

Let $\sigma \subset TM$ is a 2-plane, $\sigma = u \wedge v$, $|u| = |v| = 1$, $u \cdot v = 0$.

$$u = u^i e_i \quad v = v^i e_i$$

$$\sec(\sigma) = \sec(u, v) = R(u, v, u, v)$$

$$= u^i v^j u^k v^l R(e_i, e_j, e_k, e_l)$$

$$= u^i v^j u^k v^l F^4 R_{ijkl}$$

$$= -u^i v^j u^k v^l (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$$

$$= -\sum_{i,j} (u^i)^2 (v^j)^2 + \sum_{i,j} u^i v^i v^j u^j$$

$$= -1$$

curvature of S^2

$$f(\theta, \varphi) = (\sin \theta, \cos \theta \sin \varphi, \cos \theta \cos \varphi)$$

$$f(0,0) = (0, 0, 1) =: N$$

$$\frac{\partial f}{\partial \theta} = (\cos \theta, -\sin \theta \sin \varphi, -\sin \theta \cos \varphi) \quad \frac{\partial f}{\partial \theta}(0,0) = (1, 0, 0)$$

$$\frac{\partial f}{\partial \varphi} = (0, \cos \theta \cos \varphi, -\cos \theta \sin \varphi) \quad \frac{\partial f}{\partial \varphi}(0,0) = (0, 1, 0)$$

$$T_N S^2 = \text{span} \left\{ \frac{\partial f}{\partial \theta}(0,0), \frac{\partial f}{\partial \varphi}(0,0) \right\} = e_1, e_2$$

$$X = \frac{\partial}{\partial \theta}$$

$$\begin{cases} \frac{\partial X}{\partial \theta} = (-\sin \theta, -\cos \theta \sin \varphi, -\cos \theta \cos \varphi)^T = 0 \\ \frac{\partial^2 X}{\partial \varphi \partial \theta} = 0 \end{cases}$$

$$\begin{cases} \frac{\partial X}{\partial \varphi} = (0, -\sin \theta \cos \varphi, \cos \theta \sin \varphi)^T \\ \quad = (0, -\sin \theta \cos \varphi, \cos \theta \sin \varphi) \\ \frac{\partial^2 X}{\partial \theta \partial \varphi} = (0, -\cos \theta \cos \varphi, -\sin \theta \sin \varphi)^T \\ \quad = (0, -\cos \theta \cos \varphi, -\sin \theta \sin \varphi) \end{cases}$$

$$\frac{\partial^2 X}{\partial \theta \partial \varphi}(0,0) = (0, -1, 0) = -e_2$$

$$\Rightarrow \frac{\partial^2 X}{\partial \varphi \partial \theta} - \frac{\partial^2 X}{\partial \theta \partial \varphi} = R \left(\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \varphi} \right) X.$$

$$\Rightarrow e_2 = R(e_1, e_2) e_1.$$

$$\begin{aligned} \text{Sec}(e_1, e_2) &= R(e_1, e_2, e_1, e_2) \\ &= \langle R(e_1, e_2) e_1, e_2 \rangle = 1 \end{aligned}$$

V inner product space. $\dim V = n$.

$$B: V \times V \longrightarrow \mathbb{R}.$$

bilinear operator. $\exists!$ linear operator $A: V \longrightarrow V$

$$B(u, v) = \langle Au, v \rangle \quad \forall u, v \in V.$$

$$\text{tr } B \triangleq \text{tr } A.$$

$$\text{Let } V = \text{span} \{e_i\}_{i=1}^n$$

$$g_{ij} = \langle e_i, e_j \rangle.$$

$$Ae_i = A_{ij} e_j$$

$$g^{ij} = (g_{ij})^{-1}$$

$$B_{ij} = B(e_i, e_j).$$

$$\leadsto B_{ij} = B(e_i, e_j) = \langle Ae_i, e_j \rangle$$

$$= \langle A_i^k e_k, e_j \rangle = A_i^k g_{kj}$$

$$A_i^j = B_{ik} g^{kj}$$

$$\Rightarrow \boxed{\text{tr } B = \text{tr } A = A_i^i = B_{ij} g^{ji}}$$

$T: \mathcal{V}(M) \times \mathcal{V}(M) \rightarrow C^\infty(M)$ is a tensor. Define $\text{tr } T: M \rightarrow \mathbb{R}$.

$$\text{tr } T(p) = \text{trace of } T: T_p M \times T_p M \rightarrow \mathbb{R}$$

$$p \leadsto (U, \varphi) = (U, x^1, \dots, x^n) \text{ chart.}$$

$$\text{tr } T = T_{ij} g^{ij} \in C^\infty(M).$$

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

$T: \mathcal{V}^k(M) \rightarrow C^\infty(M)$ is a tensor $k \geq 3$, $1 \leq \alpha < \beta \leq k$.

Define (张量缩并).

$$T_{(\alpha, \beta)}(X_1, \dots, \hat{X}_\alpha, \dots, \hat{X}_\beta, \dots, X_k)$$

$$= \text{tr } T(X_1, \dots, X_{\alpha-1}, \cdot, X_{\alpha+1}, \dots, X_{\beta-1}, \cdot, X_{\beta+1}, \dots, X_k).$$

$T_{(\alpha, \beta)}$ is a tensor of $(k-2)$ -order.

$$(T_{(\alpha, \beta)})_{i_1, \dots, \hat{i}_\alpha, \dots, \hat{i}_\beta, \dots, i_k} = T_{i_1, \dots, i_k} g^{i_\alpha i_\beta}.$$

R is a tensor of order 4.

$$\begin{array}{cccccc} R_{(1,2)} & R_{(1,3)} & R_{(1,4)} & R_{(1,3)} & R_{(2,4)} & R_{(3,4)} \\ \parallel & & & & & \parallel \\ 0 & & & & & 0. \end{array}$$

$$\underbrace{R_{ijkl} g^{ij}}_{\substack{\text{ij 反对称} \\ \text{ij 对称}}}$$

$$R_{(1,2)} = R_{(2,4)} = -R_{(1,4)} = -R_{(2,3)}.$$

只有一个独立.

Define $\text{Ric.} = R_{(2,4)}.$

$$(\text{Ric})_{ij} = R_{ikjl} g^{kl}.$$

Let $T_p M = \text{span} \{e_i\}_{i=1}^n$ $\langle e_i, e_j \rangle = \delta_{ij}$.

$$\begin{aligned} \text{Ric}(e_i, e_j) &= \text{trace of } R(e_i, \cdot, e_j, \cdot) \\ &= R(e_i, e_k, e_j, e_k) \cdot \delta^{ke} \\ &= \sum_k R(e_i, e_k, e_j, e_k) \end{aligned}$$

if $n=2$:

$$\text{Ric}(e_1, e_1) = R(e_1, e_2, e_1, e_2) = \text{Sec}(e_1, e_2)$$

if $n=3$:

$$\begin{aligned} \sigma &\subset T_p M \quad \sigma = e_1 \wedge e_2 \\ \begin{cases} \text{Ric}(e_1, e_1) = \sum_{k=1}^3 R(e_1, e_k, e_1, e_k) = \text{Sec}(e_1, e_2) + \text{Sec}(e_1, e_3) \\ \text{Ric}(e_2, e_2) = \text{Sec}(e_2, e_1) + \text{Sec}(e_2, e_3) \\ \text{Ric}(e_3, e_3) = \text{Sec}(e_3, e_1) + \text{Sec}(e_3, e_2) \end{cases} \end{aligned}$$

$$\leadsto \text{Sec}(e_1, e_2), \text{Sec}(e_1, e_3), \text{Sec}(e_2, e_3).$$

Def: $\text{scal} = \text{Tr}(\text{Ric})$

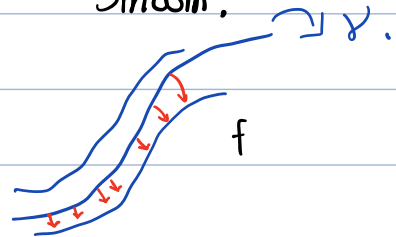
if $M \leadsto$ constant curvature k .

$$\text{scal} = n(n-1)k.$$

Jacobi field

$$\begin{cases} \gamma : [0, a] \rightarrow M \text{ smooth.} \\ f = f(s, t) : (-\varepsilon, \varepsilon) \times [0, a] \rightarrow M \text{ smooth.} \\ f_s(t) =: f(s, t) \leadsto f_0 = \gamma \end{cases}$$

- f is called the variation of γ
- $J = \frac{\partial f}{\partial s} \big|_{s=0}$: variation vector field.
- f_s : Variation curves
- If f_s is a geodesic for all s , f is a geodesic variation.



$\gamma : [0, a] \rightarrow M$ is a geodesic, f is a geodesic variation.
 $J = \frac{\partial f}{\partial s} \big|_{s=0}$.

$$f_s : \text{geodesic} \Rightarrow \frac{\partial}{\partial t} \frac{\partial}{\partial t} f_s = 0.$$

$$\leadsto 0 = \frac{\partial}{\partial s} \frac{\partial}{\partial t} \frac{\partial}{\partial t} f = \frac{\partial}{\partial t} \frac{\partial}{\partial s} \frac{\partial}{\partial t} f + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) \frac{\partial f}{\partial t}.$$

$$= \frac{\partial}{\partial t} \frac{\partial}{\partial t} \frac{\partial}{\partial s} f + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) \frac{\partial f}{\partial t}$$

Let $s=0$, we have.

$$0 = J'' + R(\gamma', J)\gamma' \quad \text{Jacobi equation.}$$

Def. Let $\gamma : [0, a] \rightarrow M$ be a geodesic. J is a smooth vector field along γ .

$$J'' + R(\gamma', J)\gamma' = 0 \quad (*).$$

If J solves $(*)$, J is called Jacobi field.

Example. $\gamma', t\gamma'(t)$.

$$J = g\gamma' \leadsto g''\gamma' + gR(\gamma', \gamma')\gamma' = 0.$$

$$\leadsto g'' = 0$$

$$\leadsto g = \alpha + \beta t$$

$$\leadsto J = \alpha\gamma' + \beta t\gamma'$$

Theorem. Let J be a smooth vector field along γ . Then J is a Jacobi field iff there is a geodesic variation $f : (-\varepsilon, \varepsilon) \times [0, a] \rightarrow M$ such that

$$J = \frac{\partial f}{\partial s} \Big|_{s=0}.$$

Proof.

Let J be a Jacobi field. Let $\alpha : (-\varepsilon, \varepsilon) \rightarrow M$

$$\alpha(0) = \gamma(0).$$

$$\alpha'(0) = J(0). \quad \text{测地线起点变化率}$$

$$\hookrightarrow J(0) = \frac{d}{ds} \Big|_{s=0} f(s, 0)$$

Let X be a vector field along α .

$$X(0) = \gamma'(0).$$

$$X'(0) = J'(0). \quad \text{初速度变化率}$$

$$\text{Let } f(s, t) = \exp_{\alpha(s)}(tX(s)).$$

$$\hookrightarrow J'(0) = \frac{\partial^2}{\partial s \partial t} f(0, 0) = \frac{d}{ds} \Big|_{s=0} \left(\frac{\partial f}{\partial t}(s, 0) \right)$$

$$f(0, t) = \exp_{\gamma(0)}(t\gamma'(0)) = \gamma(t).$$

$$\underbrace{f_s(0)}_{\gamma'(0)}$$

$$K =: \frac{\partial}{\partial s} f \Big|_{s=0}$$

$$K(0) = \frac{\partial}{\partial s} f(0,0) = \frac{\partial}{\partial s} \Big|_{s=0} f(s,0) = \alpha'(0) = J(0).$$

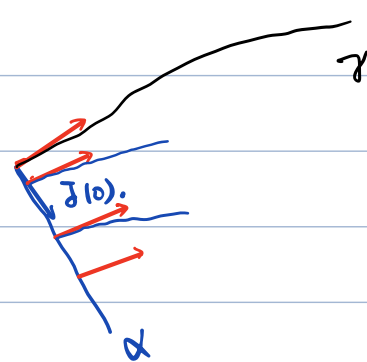
$$K'(0) = \frac{\partial'}{\partial t} f(0,0) = \frac{\partial'}{\partial s} f(0,0)$$

$$= \frac{\partial}{\partial s} \Big|_{s=0} \cdot \frac{\partial f}{\partial t}(s,0)$$

$$= \frac{\partial}{\partial s} \Big|_{s=0} X(s) = X'(0) = J'(0).$$

$$\Rightarrow K = J$$

□



$$\begin{cases} J'' + R(\gamma', J) \gamma' = 0 & \gamma(0) = p. \end{cases}$$

$$\begin{cases} J(0) = v, & J'(0) = w. & (v, w) \in T_p(M)^2 \end{cases}$$

$\mathcal{J}$: the set of solutions

$$S : T_p M \times T_p M \longrightarrow \mathcal{J} \quad \text{linear isomorphism.} \quad \dim \mathcal{J} = 2n$$

$$\mathcal{J}^\perp = \{ J \in \mathcal{J} : J(t) \perp \gamma'(t) \text{ for all } t \}.$$

$$\text{Claim: } \mathcal{J} = \mathcal{J}^\perp \oplus \text{span} \{ \gamma', t\gamma' \}. \quad \text{then } \dim \mathcal{J}^\perp = 2(n-1)$$

Proposition. Let $J \in \mathcal{J}$, Then $J \in \mathcal{J}^\perp$ iff

$$\langle J(0), \gamma'(0) \rangle = \langle J'(0), \gamma'(0) \rangle = 0$$

$$\text{Proof. } \frac{d^2}{dt^2} \langle J, \gamma' \rangle = \langle J'', \gamma' \rangle + 2 \langle J', \gamma'' \rangle + \langle J, \gamma''' \rangle$$

$$= - \langle R(\gamma', J) \gamma', \gamma' \rangle = - R(\gamma', J, \gamma', \gamma') = 0$$

$$\Rightarrow \langle J, \gamma' \rangle = at + b$$

$$\text{but } \langle J(0), \gamma'(0) \rangle = \langle J'(0), \gamma'(0) \rangle = 0, \quad \text{we have } a=b=0$$

$$\text{i.e. } \langle J, \gamma' \rangle = 0.$$

$$\text{Prop. } \mathcal{J} = \mathcal{J}^\perp \oplus \text{span} \{ \gamma', t\gamma' \}.$$

proof. It suffices to check that $\dim \mathcal{J}^\perp = 2(n-1)$. Indeed,

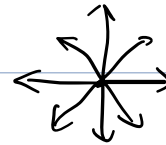
$$\mathcal{J}^\perp = \{ J \in \mathcal{J} ; J(0) \perp \gamma'(0), J'(0) \perp \gamma'(0) \}$$

$$= S(\{\gamma'(0)\}^\perp \times \{\gamma'(0)\}^\perp)$$

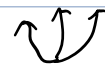
$$\text{It follows that } \dim \mathcal{J}^\perp = \dim \{\gamma'(0)\}^\perp \times \{\gamma'(0)\}^\perp = 2(n-1)$$

$\bar{J}$: spreading velocity of geodesic $J(0) = 0 \rightsquigarrow$

$|\bar{J}|$: spreading rate of geodesic



R^2



S^2



H^2

Theorem. Let $\gamma: [0, a] \rightarrow M$ be a geodesic with

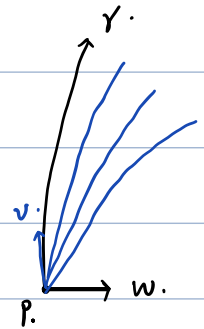
$$\gamma(0) = p \quad \gamma'(0) = v \quad |v| = 1$$

Let J be a Jacobi field along γ with

$$J(0) = 0 \quad J'(0) = w, \quad |w| = 1, \quad \langle v, w \rangle = 0$$

Then

$$|J(t)| = t - \frac{1}{6} \sec(v, w) t^3 + o(t^3) \quad \text{as } t \rightarrow 0^+$$



Proof. $\frac{d}{dt} \Big|_0 |J|^2 = 2 \langle J', J \rangle \Big|_0 = 0. \quad (\langle J, w \rangle = 0)$

$$\frac{d^2}{dt^2} \Big|_0 |J|^2 = 2 \langle J'', J \rangle \Big|_0 + 2 \langle J', J' \rangle \Big|_0 = 2 |w|^2 = 2$$

$$\frac{d^3}{dt^3} \Big|_0 |J|^2 = 4 \langle J'', J' \rangle \Big|_0 + 2 \langle J''', J \rangle \Big|_0 + 2 \langle J'', J' \rangle \Big|_0$$

$$= 6 \langle J'', J' \rangle \Big|_0 = -6 \langle R(\gamma', J) \gamma', J' \rangle \Big|_0$$

$$= -6 \langle R(v, 0) v, w \rangle = 0$$

$$\frac{d^4}{dt^4} |J|^2 = 6 \langle J'', J'' \rangle \Big|_0 + 8 \langle J''', J' \rangle \Big|_0 + 2 \langle J^{(4)}, J \rangle \Big|_0$$

$$= 8 \langle J''', J' \rangle \Big|_0$$

$$J'''(0) = - \frac{d}{dt} \Big|_0 R(\gamma', J) \gamma' = - R(\gamma'(0), J'(0)) \gamma'(0)$$

$$\Rightarrow \frac{d^4}{dt^4} |J|^2 = -8 R(v, w, v, w) = -8 \sec(v, w)$$

thus, $|J|^2 = t^2 - \frac{8}{4!} \sec(v, w) t^4 + o(t^4)$

$$= t^2 - \frac{1}{3} \sec(v, w) t^4 + o(t^5)$$

$$\Rightarrow \frac{|J|^2}{t^2} = 1 - \frac{1}{3} \sec(v, w) t^2 + o(t^3)$$

$$\Rightarrow \frac{|J|}{t} = 1 - \frac{1}{6} \sec(v, w) t^2 + o(t^3)$$

$$\Rightarrow |J| = t - \frac{1}{6} \sec(v, w) t^3 + o(t^3)$$

Let X, Y, Z be smooth vector fields along γ . $X(0) = 0$. Then.

$$\frac{d}{dt} T(X, Y, Z) = T(Y, X, Z) \Big|_0$$

Where T is a 3-tensor

Proof: Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart near $\gamma(t_0)$. then let

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^i \frac{\partial}{\partial x^i}, \quad Z = Z^i \frac{\partial}{\partial x^i}$$

$$\begin{aligned} \frac{d}{dt}\bigg|_{t_0} T(Y, X, Z) &= \frac{d}{dt}\bigg|_{t_0} X^i T(Y, \frac{\partial}{\partial x^i}, Z) = \dot{X}^i(t_0) T(Y(t_0), \frac{\partial}{\partial x^i}\bigg|_{\gamma(t_0)}, Z(t_0)) \\ &= T(Y(t_0), X(t_0), Z(t_0)) \end{aligned}$$

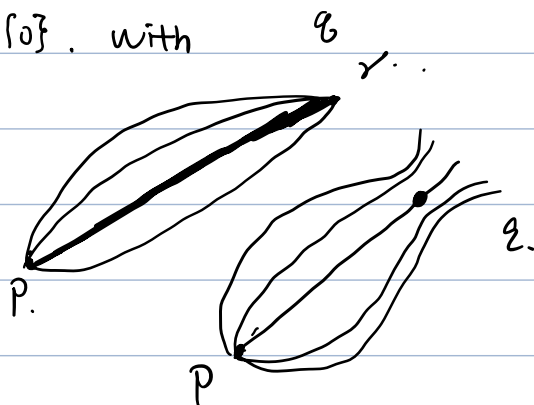
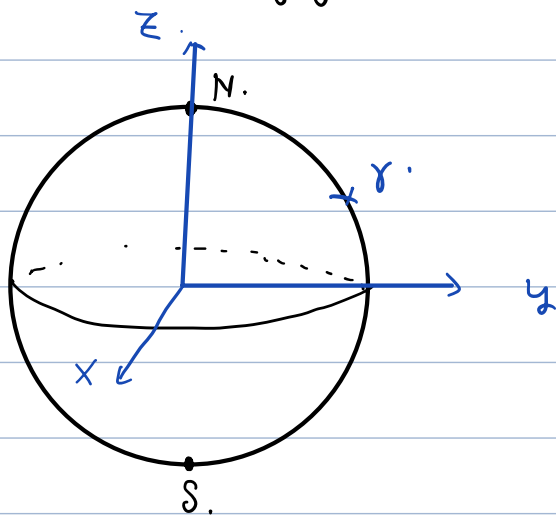
Conjugate point

$\gamma: [0, a] \rightarrow M$ is a geodesic, $\gamma(t_0) = p$.

Def. Let $q = \gamma(t_0)$. $0 < t_0 \leq a$. If $\exists J \in \mathcal{J} \setminus \{0\}$ with

$$J(t_0) = 0, \quad J'(t_0) \neq 0$$

We say q is a conjugate point of p .



$S \rightsquigarrow$ conjugate point of N .

$$\gamma(t) = u \cos t + v \sin t.$$

Where, $u = (0, 0, 1)$ $v = (0, 1, 0)$

Let $f(s, t) = \exp_N(t \alpha(s))$

Where $\alpha: (-\epsilon, \epsilon) \rightarrow T_p S^2$, $\alpha(t_0) = \gamma'(t_0)$. $\alpha'(t_0) =: w$

$$\mathcal{J}_0 = \{J \in \mathcal{J} : J(0) = 0\} = \mathcal{S}(\{0\} \times T_p M) \leadsto \dim \mathcal{J}_0 = n$$

$$\mathcal{J}_{0,0} = \{J \in \mathcal{J} : J(0) = 0, J(t_0) = 0\}. \quad \gamma(t_0) \text{ conjugate point of } p \text{ implies } \dim \mathcal{J}_{0,0} > 0.$$

$$\text{mul}(q) =: \dim \mathcal{J}_{0,0} \quad \text{重数.}$$

$$t\gamma'(t) \in \mathcal{J}_0 \setminus \mathcal{J}_{0,0} \leadsto \dim \mathcal{J}_{0,0} \leq n-1.$$

Definition. Let $f: M_1 \rightarrow M_2$ be smooth, $q \in M_1$. If df_q is not a surjection, we say q is a critical point of f .

$$\text{If } \dim M_1 = \dim M_2 \quad df_p \text{ is not surjection} \iff df_p \text{ is not injection.} \\ \iff \dim \ker(df_p) > 0$$

Let $\gamma: [0, a] \rightarrow M$ be a geodesic

$$p = \gamma(0) \quad q = \gamma(t_0)$$

$$\text{then.} \quad \gamma(t) = \exp_p(t\gamma'(0)) \quad q = \exp_p(t_0\gamma'(0)).$$

$$v_0 =: t\gamma'(0) \Rightarrow q = \exp_p(v_0).$$

Proposition. q is a conjugate point iff v_0 is a critical point of $\exp_p$. Moreover, if q is a critical point of $\exp_p$ $\text{mul}(q) = \dim \ker(d\exp_p)_{v_0}$.

Proof. If $J \in \mathcal{J}_0$, then.

$$J = \frac{d}{ds}\bigg|_{s=0} \exp_p(t(\gamma'(0) + sJ'(0))) = \frac{\partial}{\partial s} f(s, t)$$

It suffices to check that $\dim \mathcal{J}_{0,0} = \dim \ker(d\exp_p)_{v_0}$

$$J(t_0) = \frac{d}{ds}\bigg|_{s=0} \exp_p(t_0(\gamma'(0) + sJ'(0))) \\ = (d\exp_p)_{(t_0\gamma'(0))} (t_0 J'(0)) = t_0 (d\exp_p)_{v_0} (J'(0))$$

It follows that:

$$J \in \mathcal{J}_{0,0} \iff (d\exp_p)_{v_0} (J'(0)) = 0 \\ \iff J'(0) \in \ker(d\exp_p)_{v_0}.$$

$$\mathcal{J}_{0,0} = \mathcal{S}(\{0\} \times \ker(d\exp_p)_{v_0})$$

Jacobi field on a manifold with constant curvature.

Def. Let $\alpha : [a, b] \rightarrow M$ be a smooth curve, X is a vector field along α . If $\frac{dX}{dt} = 0$.

We say X is a parallel vector field along α .

$$\begin{cases} \frac{dX}{dt} = 0 \\ X(0) = X \end{cases} \quad (*) \quad \mathcal{X} = \{ \text{solution of } (*) \}.$$

Proposition. Let $X, Y \in \mathcal{X}$, then

$$\langle X, Y \rangle = \langle X(0), Y(0) \rangle$$

Def. Let $\gamma : [0, a] \rightarrow M$ be a geodesic, $\gamma(0) = p$

$$T_p M = \text{span} \{ e_i \}_{i=1}^n \quad \langle e_i, e_j \rangle = \delta_{ij}.$$

Let E_i be the parallel vector fields along γ with $E_i(0) = e_i$. then

$$\langle E_i(t), E_j(t) \rangle = \delta_{ij} \quad \forall t \in [0, a]$$

$\{E_j\}_{j=1}^n$: geodesic frame.

M : constant curvature k .

$$R(x, y, u, v) = k(\langle x, u \rangle \langle y, v \rangle - \langle x, v \rangle \langle y, u \rangle)$$

$$R(x, y)u = k(\langle x, u \rangle y - \langle y, u \rangle x)$$

Given $\gamma : [a, b] \rightarrow M$ geodesic, $\gamma(0) = p$, $|\gamma'| = 1$

Let $\{E_i\}$ be a geodesic frame along γ .

$$E_n(0) = \gamma'(0) \Rightarrow E_n = \gamma'$$

Let $J \in \mathcal{J}$. $J = \sum \alpha_i E_i$

$$0 = J'' + R(\gamma', J)\gamma'$$

$$= \sum \alpha_i'' E_i + k(\langle \gamma', \gamma' \rangle J - \langle J, \gamma' \rangle \gamma')$$

$$= \sum \alpha_i'' E_i + k(\sum \alpha_i E_i - \alpha_n E_n)$$

$$= \sum_{i=1}^{n-1} (\alpha_i'' + k\alpha_i) E_i + \alpha_n'' E_n$$

$$\Rightarrow \begin{cases} \alpha_i'' + k\alpha_i = 0 & 1 \leq i \leq n-1 \\ \alpha_n'' = 0 \end{cases}$$

$$\alpha_i = \begin{cases} a_i + b_i t & k=0 \\ a_i \cos(\sqrt{k}t) + b_i \sin(\sqrt{k}t) & k>0. \\ a_i \cosh(\sqrt{-k}t) + b_i \sinh(\sqrt{-k}t) & k<0. \end{cases}$$

$$\alpha_n = a_n + b_n t$$

$$J(t) = \sum_{i=1}^{n-1} a_i E_i + \sum_{i=1}^{n-1} a_i t E_i + a_n \gamma'(t) + b_n t \gamma'(t)$$

$J(0)=0$. 部分.

$\hookrightarrow J'(0)=0$ 部分

$$k=0.$$

$$J(t) = \sum_{i=1}^{n-1} a_i \cos \sqrt{k}t E_i + \sum_{i=1}^{n-1} b_i \sin \sqrt{k}t E_i + a_n \gamma'(t) + b_n t \gamma'(t).$$

$k>0.$

$$\mathcal{M} = S^n$$

$$\gamma: [0, 2\pi] \rightarrow \mathcal{M}. \text{ geodesic. } \gamma(0)=p. \quad |\gamma'|=1 \quad \gamma(\pi)=q.$$

$$J = \sum_{i=1}^{n-1} a_i \cos t E_i + \sum_{i=1}^{n-1} b_i \sin t E_i + a_n \gamma' + b_n t \gamma'$$

$$\text{Let } a_i = 0 \text{ for all } n, \text{ then } J \in \mathcal{J}_0. \text{ Let}$$

$$J = \sum_{i=1}^{n-1} b_i \sin t E_i \Rightarrow J(\pi) = 0.$$

thus q is the unique conjugate point of p .

$$\text{mul}(q) = \dim \mathcal{J}_{00} = n-1$$