$U \subset \mathbb{R}^n$ bounded and open.

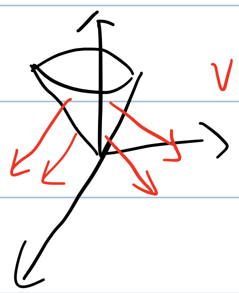
$\boxed{\text{THM (Gauss-Green theorem)}}$ $\mathbb{R}^n$

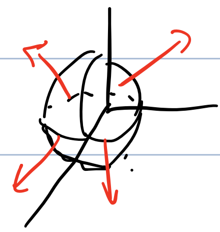
$u \in C^1(\bar{U})$, then

$$\int_U \underbrace{u_{x_i}}_{\frac{\partial}{\partial x_i}u} dx = \int_{\partial U} u \nu^i dS. \qquad \text{☆}$$

$\nu = (\nu^1, \cdots \nu^n)$ 沿着 ∂U 在 x 处的单位外法向向量.

例: $U \sim$ 球





THM (分部积分). 设 $u, v \in C^1(\bar{U})$, 则.

$$\int_U u_{x_i} v \, dx = -\int_U u v_{x_i} \, dx + \int_{\partial U} u v \nu^i dS.$$

Proof: $\displaystyle\int_U (uv)_{x_i} dx = \int_{\partial U} uv \nu^i dS.$

$$(uv)_{x_i} = \frac{\partial}{\partial x_i}(uv) = \left(\frac{\partial}{\partial x_i}u\right)v + u\frac{\partial}{\partial x_i}v.$$

$$= u_{x_i}v + u v_{x_i}$$

$$\int_U u_{x_i} v \, dx + \int u v_{x_i} dx = \int_{\partial U} uv\nu^i dS.$$

THM (Green formula). 令 $u, v \in C^2(\bar{U})$, 则. $\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g. \end{cases}$

① $\int_U \Delta u \, dx = \int_{\partial U} \frac{\partial u}{\partial \nu} \, dS.$ → Laplace.

$$\left[\begin{array}{ll} \Delta u = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u & \frac{\partial u}{\partial \nu} = \nabla u \cdot \nu \\ \nabla u = \left(\frac{\partial}{\partial x_1} u, \cdots \frac{\partial}{\partial x_n} u \right). & \end{array} \right]$$

Proof: $\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} (u_{x_i})_{x_i}$

$$\sum \int_U (u_{x_i})_{x_i} \, dx = \sum \int_{\partial U} u_{x_i} \nu^i \, dS \quad \text{(Gauss-Green)}$$

$$\| $$

$$\int_U \Delta u \, dx = \int_{\partial U} \nabla u \cdot \nu \, dS = \int_{\partial U} \frac{\partial u}{\partial \nu} \, dS.$$

② $\int_U \nabla u \cdot \nabla v \, dx = -\int_U u \Delta v \, dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u \, dS.$

$\nabla u \cdot \nabla v = \sum u_{x_i} v_{x_i}$

$u \sum (v_{x_i})_{x_i}$

Proof: 对 $u_{x_i} v_{x_i}$ 用分部积分

$$\int_U u_{x_i} v_{x_i} \, dx = -\int_U u \, v_{x_i x_i} + \int_{\partial U} u \, v_{x_i} \nu^i \, dS.$$

$\nabla v \cdot \nu = \sum v_{x_i} \nu^i$

$$\Rightarrow \int_U \nabla u \cdot \nabla v \, dx = -\int_U u \Delta v + \int_{\partial U} u \frac{\partial v}{\partial \nu} \, dS.$$

③. $$\int_U u \Delta v - v \Delta u \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \, dS.$$ ☆

Green 函数

Proof:

$$\int_U (u \Delta v + \nabla u \cdot \nabla v) \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} \, dS.$$

$$\int_U (v \Delta u + \nabla v \cdot \nabla u) \, dx = \int_{\partial U} v \frac{\partial u}{\partial \nu} \, dS.$$

$$\Rightarrow \int_U (u \Delta v - v \Delta u) \, dx = \int_{\partial U} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS.$$

THM (Polar coordinates). 波方程，球面平均

①. $f: \mathbb{R}^n \to \mathbb{R}$ 连续，$f \in L^1(\mathbb{R}^n)$,

$$\left[\int_{\mathbb{R}^n} |f(x)| \, dx < \infty. \right].$$

则.

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_0^\infty \left(\int_{\partial B(x_0, r)} f(y) \, dS(y) \right) dr.$$

$$\int_{\mathbb{R}^2} f(x) \, dx = \int_0^\infty \left(\int_0^{2\pi} f(r\cos\theta, r\sin\theta) \, r \, d\theta \right) dr$$

② $\frac{d}{dr}\left(\int_{B(x_0,r)} f(x)\,dx\right) = \int_{\partial B(x_0,r)} f\,dS$

$$\int_{B(x_0,r)} f(x)\,dx = \int_0^r \left(\int_{\partial B(0,s)} f(x)\,dS(x)\right) dr.$$

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x) \qquad \Big\downarrow \frac{d}{dr}$$

$$\int_{\partial B(0,r)} f(x)\,dS(x).$$

前置知识

① 变分问题 (Chapter 1)

实际 → 优化问题 → 得到 PDE/ODE ~ 求解

例：(第1章习题12).

求解变分问题：$u \in M = \{ y \in C^1[0,1] : y(1) = 0 \}$ s.t.

$$J(u) = \min_{y \in M} J(y)$$

其中 $J(y) = \frac{1}{2}\int_0^1 (y'(x))^2\,dx - 2\int_0^1 y(x)\,dx - y(0).$

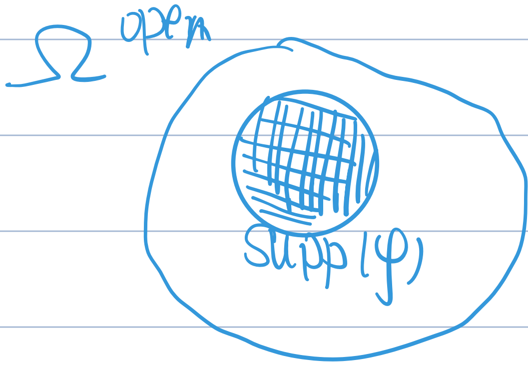
Remark: $J(\cdot)$ 自变量是函数. ☆.

$J: M \to \mathbb{R}.$

Lemma (P17) $\Omega \subset \mathbb{R}^2$ 是区域 (连通开集), $f: \Omega \to \mathbb{R}$ 连续. 若

$$\int_\Omega f\varphi \, dx\,dy = 0 \qquad \forall \varphi \in C_0^\infty(\Omega)$$

在Ω内有紧支撑



$$\varphi(x) = \begin{cases} \frac{1}{x} & x>0 \\ 0 & x=0 \end{cases}$$

$\text{supp}(\varphi) = \overline{\{x : \varphi(x) \neq 0\}}$.

$\text{supp}(\varphi) = [0,\infty)$.

则 $f \equiv 0$ Ω.

~~Solution~~

期任.

设 $u \in M$. 满足

$$J(u) = \min_{y\in M} J(y)$$

即:

$$J(u) \leq J(y) \qquad \forall y \in M.$$

则任意给定 $\varphi \in M$，则

$$u + \varepsilon\varphi \in M = \{y \in C^1[0,1];\ y(1)=0\} \quad \forall \varepsilon \in \mathbb{R}.$$

即有:

$$J(u) \leq J(u+\varepsilon\varphi)$$

考虑: $j(\varepsilon) = J(u+\varepsilon\varphi)$，则必有:

$$j(0) \leq j(\varepsilon) \qquad \forall \varepsilon \in \mathbb{R}.$$

则又有: $\hat{j}'(0)=0 \qquad \hat{j}''(0)>0.$

设 u 为变分问题的解, 即

$$J(u) \le J(y) \quad \forall\, y \in M.$$

给定 $\varphi \in M$, 则有 ($\varphi \neq 0.$)

$$u+\varepsilon\varphi \in M \qquad \forall\, \varepsilon \in \mathbb{R}.$$

考虑:

① $\hat{j}(\varepsilon) = J(u+\varepsilon\varphi)$

$$= \frac{1}{2}\int_0^1 (u'+\varepsilon\varphi')^2 dx - 2\int_0^1 u+\varepsilon\varphi\, dx - u(0)-\varepsilon\varphi(0).$$

$(u')^2+2\varepsilon u'\varphi'+\varepsilon^2(\varphi')^2.$

② 求导.

$$\hat{j}'(\varepsilon) = \int_0^1 u'\varphi' + \varepsilon(\varphi')^2 dx - 2\int_0^1 \varphi\, dx - \varphi(0).$$

$$\hat{j}''(\varepsilon) = \int_0^1 (\varphi')^2 dx$$

$$\Rightarrow \hat{j}'(0) = \int_0^1 u'\varphi' dx - 2\int_0^1 \varphi dx - \varphi(0).$$

$$\hat{j}''(0) = \int_0^1 (\varphi')^2 dx > 0.$$

③ 令 $\hat{j}'(0)=0$, 去找 φ 满足的条件 (通常使用分部积分).

$\to \varphi.$

$$0 = \int_0^1 u'\varphi' dx - 2\int_0^1 \varphi\, dx - \varphi(0)$$

$$= -\int_0^1 u''\varphi\, dx + u'\varphi\Big|_0^1 - 2\int_0^1 \varphi dx - \varphi(0)$$

$$= -\int_0^1 (u''+2)\varphi\,dx - (1+u'(0))\varphi(0)$$

i.e. $\forall \varphi \in M = \{ \varphi \in C^1[0,1];\ \varphi(1)=0 \}$ 都有

$$-\int_0^1 (u''+2)\varphi\,dx - \underline{(1+u'(0))\varphi(0)} = 0. \quad \star$$

任取 $\varphi \in C_0^\infty \subset M$ 且 $\varphi(0)=\varphi(1)=0$. 则.

$$\int_0^1 (u''+2)\varphi\,dx = 0. \qquad \forall \varphi \in C_0^\infty$$

由引理: $u''+2=0$

$\Longrightarrow \int_0^1 (u''+2)\varphi = 0.\quad \forall \varphi \in M.$

代入 $(\star)$ 式后:

$$\boxed{(1+u'(0))\varphi(0)=0.} \qquad \forall \varphi \in M.$$

由 φ 选取任意性:

$$1+u'(0)=0.$$

又 $u \in M \Rightarrow u(1)=0.$

综上:

$$\left\{\begin{array}{l} u''+2=0. \\ u(1)=0 \\ u'(0)=-1 \end{array}\right\} \xrightarrow{\text{解 ODE}} u(x) = -x^2-x+2.$$

习题 Chapter 1 13, 14.

特征线法 (一阶 PDE)

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t} + \alpha(x,t)\dfrac{\partial u}{\partial x} + u = \beta(x,t) \\ u(x,0) = f(x). \end{array}\right. \quad \star$$

思想：x 看成 t 的函数

特征线方程 $\begin{cases} \dfrac{dx}{dt} = \alpha(x,t) \\ x(0) = C \end{cases}$ ODE.

$x(t)$ $\rightsquigarrow$ $v(t) = u(x(t),t)$ 满足

$$\frac{dv}{dt} = \frac{d\,u(x(t),t)}{dt} = \frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

$$= \frac{\partial u}{\partial t} + \alpha(x,t)\frac{\partial u}{\partial x}.$$

☆转化为：$\begin{cases} \dfrac{dv}{dt} + v = \beta(x(t),t). \\ v(0) = u(x(0),0) = f(x(0)). \end{cases}$ ODE.

例：求 Cauchy 问题的解

$$\star\ \begin{cases} \dfrac{\partial u}{\partial t} + (x+t)\dfrac{\partial u}{\partial x} + u = x & (x,t) \in \mathbb{R} \times [0,\infty) \\ u(x,0) = x. \end{cases}$$

① 求特征线

$$\begin{cases} \dfrac{dx}{dt} = x+t \\ x(0) = C \end{cases}$$

$$x(t) = e^t(1+C) - (1+t)$$ (与C有关)

② 令 $v(t) = u(x(t),t)$，转化 (★) 为 ODE.

$$\rightarrow \begin{cases} \dfrac{dv}{dt} + v = x(t) = e^t(1+C) - (1+t). \\ v(0) = u(x(0),0) = x(0) = C. \end{cases}$$

$$v(t) = \frac{1}{2}(C+1)e^t + \frac{1}{2}(C-1)e^{-t} - t$$

③ 反解 u.

反解 (x,t)，代换 C

$$u(x,t) = \frac{1}{2}e^{-2t}(x+t+1) - e^{-t} + \frac{1}{2}(x-t+1)$$

波动方程

$$\begin{cases} \left(\left(\frac{\partial}{\partial t}\right)^2 - a^2 \left(\frac{\partial}{\partial x}\right)^2 \right) u = 0 \\ u(x,0) = 0. \\ u_t(x,0) = \psi(x). \end{cases}$$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\frac{\partial}{\partial t}$$

分解.

☆. $\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}$

$=: v(x,t)$

$$\rightsquigarrow \begin{cases} \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)\boxed{\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)u} = 0. \\ u(x,0) = 0. \\ u_t(x,0) = \psi(x). \end{cases}$$

$$\rightsquigarrow \begin{cases} \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right) v = 0. \\ v(x,0) = \frac{\partial}{\partial t}u(x,0) - a\frac{\partial}{\partial x}u(x,0). \\ \quad = \psi(x). \end{cases}$$

$$\begin{cases} \left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right) u = v. \\ u(x,0) = 0 \end{cases}$$

课本 P103 T16.

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} + 1\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right).$$

行波法解波动方程.

$$\left(\frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial^2}{\partial x^2}\right) u = 0. \xrightarrow[(t,x)\mapsto(\xi,\eta)]{\text{变量替换}} u_{\xi\eta} = 0.$$

$$\begin{cases} \xi = ax + bt \\ \eta = cx + dt \end{cases} \Longrightarrow \begin{cases} \frac{\partial}{\partial x} = a\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta} + \\ \frac{\partial}{\partial t} = b\frac{\partial}{\partial \xi} + d\frac{\partial}{\partial \eta}. \end{cases}$$

↓ 保留.

$$\left(\frac{\partial}{\partial x}\right)^2 = a^2\left(\frac{\partial}{\partial \xi}\right)^2 + 2ac\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} + c^2\left(\frac{\partial}{\partial \eta}\right)^2.$$

$$\left(\frac{\partial}{\partial t}\right)^2 = b^2\left(\frac{\partial}{\partial \xi}\right)^2 + 2bd\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} + d^2\left(\frac{\partial}{\partial \eta}\right)^2.$$

$$\frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial}{\partial x^2}.$$

$$\begin{cases} b^2 - \beta^2 a^2 = 0. \quad d^2 - \beta^2 c^2 = 0. \\ 2bd - \beta^2 \cdot 2ac \neq 0 \end{cases}$$

取 $a = c = 1$，则 $b = \beta$ $\quad c = -\beta$.

$$\rightsquigarrow \begin{cases} \xi = x \pm \beta t \\ \eta = x - \beta t. \end{cases}$$

$$\rightsquigarrow \quad u_{\xi\eta} = 0. \Rightarrow u(x(\xi,\eta), t(\xi,\eta))$$

$$= F(\xi) + G(\eta).$$

$$\Rightarrow u(x,t) = F(x+\beta t) + G(x-\beta t)$$

例：
$$\begin{cases} u_{tt} - \rho u_{xx} = 0. \\ u|_{x=3t} = a(x) \\ u|_{x=-3t} = b(x) \\ a(0) = b(0) \end{cases}$$

Solution：
$$\begin{cases} \xi = t + 3x \\ \eta = t - 3x \end{cases} \Longrightarrow u_{\xi\eta} = 0 \Rightarrow u = F(\xi) + G(\eta)$$

$$= F(x+3t) + G(x-3t)$$

$$u|_{x=3t} = F(2x) + G(0) = a(x) \quad ①$$

$u|_{x=-3t} = F(0) + G(2x) = b(x).$ ②

①：令 $x=0$ 有 $F(0)+G(0)=a(0)$

② 令 $x=0$ 有 $F(0)+G(0)=b(0)$

由①： $F(x)=a(\frac{x}{2})-G(0)$

由②： $G(x)=b(\frac{x}{2})-F(0)$

$\Rightarrow u(x,t)=F(x+3t)+G(x-3t)$

$$= a\left(\frac{x+3t}{2}\right)-G(0)+b\left(\frac{x-3t}{2}\right)-F(0)$$

$$= a\left(\frac{x+3t}{2}\right)+b\left(\frac{x-3t}{2}\right)-a(0).$$

ϕ ϕ

例(P101. T7)

Cauchy 问题

$$\begin{cases} u_{tt}-u_{xx}=6(x+t) \\ u|_{t=x}=0, \quad u_t|_{t=x}=u_1(x) \end{cases}$$

有解 $\Longleftrightarrow$ $u_1(x)=3x^2+C$

Proof: $\begin{cases} \xi = x+t \\ \eta = x-t \end{cases}$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}. \\ \frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}. \end{cases}$$

$$\left(\frac{\partial}{\partial t}\right)^2-\left(\frac{\partial}{\partial x}\right)^2=-4\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}.$$

$\Rightarrow u_{\xi\eta}=-\frac{3}{2}\xi$

$\Rightarrow u_\eta=-\frac{3}{4}\xi^2+G(\eta)$

$\Rightarrow u=-\frac{3}{4}\xi^2\eta+F(\eta)$ ← 任意的.

$\Rightarrow \quad u(x,t) = \frac{-3}{4}(x+t)^2(x-t) + F(x-t).$

$u|_{x=t} = F(0) = 0.$

$u_t|_{x=t} = \frac{3}{4}(x+t)^2 - 6(x+t)(x-t) - F'(x-t)$

$= 3x^2 - F'(0).$

$= u_1(x).$

$\Rightarrow \quad u_1(x) = 3x^2 - F'(0)$

例 (P101. T8).

$$\left\{\begin{array}{l} u_{tt} - a^2(u_{xx}+u_{yy}+u_{zz}) = 0. \\ u|_{t=0} = f(x) + g(y) \\ u_t|_{t=0} = \varphi(y) + \psi(z). \end{array}\right\} \text{三维.}$$

$$\left\{\begin{array}{l} (u_1)_{tt} - a^2(u_1)_{xx} = 0. \\ u_1|_{t=0} = f(x). \\ (u_1)_t|_{t=0} = 0. \end{array}\right. \qquad \left\{\begin{array}{l} (u_2)_{tt} - a^2(u_2)_{yy} = 0 \\ u_2|_{t=0} = g(y) \\ (u_2)_t|_{t=0} = \varphi(y) \end{array}\right.$$

$$\left\{\begin{array}{l} (u_3)_{tt} - a^2 (u_3)_{zz} = 0 \\ u_3|_{t=0} = 0 \\ (u_3)_t|_{t=0} = \psi(z). \end{array}\right. \qquad \text{一维.}$$

$u_1 + u_2 + u_3 = u.$

$u_{tt} - a^2(u_{xx} + u_{yy} + u_{zz})$

$= \left((u_1)_{tt} - a^2(u_1)_{xx}\right) + (u_2)_{tt} - a^2(u_2)_{yy} + (u_3)_{tt} - a^2(u_3)_{zz}$

$= 0.$

$$u|_{t=0} = (u_1 + u_2 + u_3)|_{t=0} = f(x) + g(y).$$

$$(u_t)|_{t=0} = (u_1 + u_2 + u_3)_t|_{t=0} = \varphi(y) + \psi(z).$$