P102, 12.

证明半无界问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t) & (x,t)\in[0,\infty)\times(0,\infty) \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x) & 0\leq x<\infty \\ u|_{x=0} = \mu(t). & t\geq 0. \end{cases}$$

解唯一.

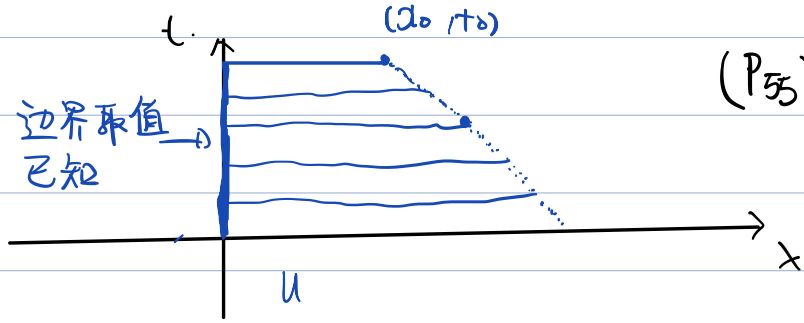
Proof: 只需证

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \\ u|_{t=0} = u_t|_{t=0} = 0 \\ u|_{x=0} = 0 \end{cases}$$

只有零解.

任意给定 $(x_0, t_0)\in[0,+\infty)\times(0,+\infty)$. 定义能量

$$E(t) = \frac{1}{2}\int_0^{x_0+a(t_0-t)} u_t^2 + a^2 u_x^2 \, dx$$



(P55)

① 对t求导

$$\frac{dE}{dt} = -\frac{a}{2}\left(u_t^2 + a^2 u_x^2\right)\Big|_{x_0+a(t_0-t)} + \int_0^{x_0+a(t_0-t)} \left(u_t u_{tt} + a^2 u_x u_{xt}\right) dx$$

$$= -\frac{a}{2}\left(u_t^2 + a^2 u_x^2\right)\Big|_{x_0+a(t_0-t)}.$$

$$+\int_0^{x_0+a(t_0-t)}(u_tu_{tt}-a^2u_tu_{xx})\,dx$$

$$+\quad a^2u_xu_t\Big|_0^{x_0+a(t_0-t)}$$

$u(0,t)=0 \qquad u_t(0,t)=\lim_{t\to 0^+}\frac{u(0,t)-u(0,0)}{t}=0$

$$=\int_0^{x_0+a(t_0-t)}u_t(u_{tt}-a^2u_{xx})\,dx.$$

$$+\left(a^2u_xu_t-\frac{a}{2}(u_t^2+a^2u_x^2)\right)\Big|_{x_0+a(t_0-t)}$$

$[\ a^2u_xu_t=\sqrt{a}\,u_t\cdot\sqrt{a^3}\,u_x\le\frac{1}{2}(au_t^2+a^3u_x^2).$

$$\le 0.$$

即 $\frac{dE}{dt}\le 0.$

② 对 $\frac{dE}{dt}$ 积分

$$0\ge\int_0^{\tau}\frac{dE}{dt}\,dt=E(\tau)-E(0) \qquad 0\le\tau\le t_0.$$

即 $0\le E(\tau)\le E(0)=\frac{1}{2}\int_0^{x_0+at_0}u_t^2(x,0)+a^2u_x^2(x,0)\,dx$

$$=0.$$

故 $E(\tau)=0 \quad \forall\,\tau\in[0,t_0]$

$u_t=u_x=0. \longrightarrow u$ 在  上是常值，但 $u|_{x=0}=0$

故 $u=0$ on  (x_0,t_0)，但 (x_0,t_0) 是任取的，故我们可说在

$(0,\infty)\times(0,\infty)$ 上 $u=0.$

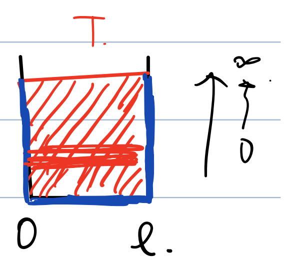
类似于 (x_0,t_0).

P83: 混合问题能量不等式

设 $u\in C^1(\overline{Q_T})\cap C^2(Q_T)$. $\quad Q_T=(0,l)\times(0,T).$

是下述 PDE 的解

$$\begin{cases}u_{tt}-a^2u_{xx}=f(x,t)\quad 0<x<l,\ t>0.\\ u(0,t)=u(l,t)=0. \quad t>0.\end{cases}$$



$$\begin{cases} u(x,0)=\varphi(x) & 0\leq x\leq l. \\ u_t(x,0)=\psi(x) & 0\leq x\leq l. \end{cases}$$

①定义能量.

$$E(t) = \frac{1}{2}\int_0^l u_t^2 + a^2 u_x^2 \, dx \qquad 0\leq t\leq T.$$

② 求导

$$\frac{dE}{dt} = \int_0^l u_t u_{tt} + a^2 u_x u_{xt} \, dx$$
$$= \int_0^l \underline{u_t u_{tt} - a^2 u_{xx} u_t} \, dx + (a^2 u_x u_t)\Big|_0^l.$$
$$= (a^2 u_x u_t)\Big|_0^l + \int_0^l u_t f(x,t)\, dx.$$

③ 对 $\frac{dE}{dt}$ 积分.

$$\underline{E(\tau) - E(0)} = \int_0^\tau \int_0^l \boxed{u_t f(x,t)} \, dx dt + \int_0^\tau (a^2 u_x u_t)\Big|_0^l dt$$
$$\leq \underline{\frac{1}{2}\int_0^\tau \int_0^l u_t^2 + a^2 u_x^2 \, dx dt} + \frac{1}{2}\int_0^\tau \int_0^l f^2 \, dx dt$$
$$+ \int_0^\tau a^2 u_x u_t\Big|_0^l dt \sim\!\!\to 0.$$

$u(0,t)=u(l,t)=0.$

$$\Omega(t) = \int_0^t E(\tau)\, d\tau.$$

$$\Longrightarrow \quad \frac{d\Omega}{dt}(\tau) \leq \underline{E(0) + \frac{1}{2}\int_0^\tau\int_0^l f^2 \, dx dt} + \Omega(\tau). \ (※).$$

③ 由 Gronwall Inequality:

$$\tfrac{1}{2}\int_0^\tau\int_0^l u_t^2 + a^2 u_x^2 \, dx dt \ \Omega(\tau) \leq M\left(E(0) + \frac{1}{2}\int_0^\tau\int_0^l f^2 \, dx dt\right).$$
$$= M\left(\frac{1}{2}\int_0^l \underline{u_t^2(x,0)} + a^2 u_x^2(x,0)\, dx + (\quad)\right)$$
$$= \tilde{M}\left(\int_0^l \varphi^2(x) + a^2(\psi')^2 dx + \int_0^\tau\int_0^l f^2 \, dx dt\right).$$

$$\sim\!\!\to \quad \underline{\int_0^T \int_0^l u_t^2 + a^2 u_x^2 \, dx dt} \leq$$

$\Omega(t)$.

$$M\left(\underline{\int_0^l \varphi^2 + a^2(\psi')^2 dx + \int_0^T\int_0^l f^2 \, dx dt}\right)$$

由(*) $\frac{dQ}{dt}(T) \le M\left(\int_0^l \varphi^2 + a^2(\psi')^2 dx + \int_0^T\int_0^l f^2 dxdt\right)$.

热方程能量不等式 (混合问题)

$Q_T = [0,l] \times [0,T]$, $u \in C^{\underset{x}{1},\underset{t}{0}}(\bar{Q}_T) \cap C^{\underset{x}{2},\underset{t}{1}}(Q_T)$. 是下述 PDE 的解:

$$\begin{cases} \boxed{u_t - a^2 u_{xx} = f} \quad (\star) & (x,t) \in Q_T \\ u(x,0) = \varphi(x) & 0 \le x \le l. \\ u(0,t) = u(l,t) = 0. & 0 \le t \le T \end{cases}$$

Proof: 对 $\star$ 两侧同乘 u 并积分.

$$\int_0^\tau\int_0^l u_t u - a^2 u_{xx} u \, dx\, dt = \int_0^\tau\int_0^l f u \, dx\, dt.$$

$$\le \frac{1}{2}\int_0^\tau\int_0^l f^2 dxdt + \frac{1}{2}\int_0^\tau\int_0^l u^2 dxdt.$$

$$\int_0^l u_{xx} u \, dx = -\int_0^l u_x^2 dx + \underbrace{u u_x \Big|_0^l}_{u(0,t) = u(l,t)=0} (=0)$$

$$= -\int_0^l u_x^2(x,t)\, dx$$

$$\int_0^\tau u_t u \, dt = \int_0^\tau u \, du = \frac{1}{2} u^2 \Big|_0^\tau.$$

$$= \frac{1}{2}\left(u^2(x,\tau) - u^2(x,0)\right).$$

$$= \frac{1}{2}\left(u^2(x,\tau) - \varphi^2(x)\right)$$

$$= \int_0^l\int_0^\tau u_t u \, dt\, dx - a^2\int_0^\tau\int_0^l u_{xx} u \, dx\, dt.$$

$$= \frac{1}{2}\int_0^l u^2(x,\tau) - \varphi^2(x) dx + a^2\int_0^\tau\int_0^l u_x^2(x,t) dx\, dt.$$

$$\le \frac{1}{2}\int_0^\tau\int_0^l f^2 dx dt + \frac{1}{2}\int_0^\tau\int_0^l u^2 dx dt.$$

$$\Rightarrow \int_0^l u^2(x,\tau)dx + 2a^2\int_0^\tau\int_0^l u_x^2(x,t)\, dx dt \qquad (\star)$$

$$\leq F(\tau)=\int_0^l \varphi^2 dx + \int_0^\tau\int_0^l f^2 dx dt + \int_0^\tau\int_0^l u^2 dx dt.$$

$\Omega(\tau) = \int_0^\tau\int_0^l u^2 dx dt$, 则 $\frac{d\Omega}{d\tau} = \int_0^l u^2(x,\tau)dx.$

$$\frac{d}{d\tau}\Omega(\tau) \leq \frac{d}{d\tau}\Omega(\tau) + 2a^2\int_0^\tau\int_0^l u_x^2(x,t)dx dt.$$

$$\leq F(\tau) + \Omega(\tau).$$

由 Gronwell Inequality

$$\Omega(\tau) \leq MF(\tau)$$

代回(※)

$$\boxed{\sup_{0\leq t\leq T}\int_0^l u^2(x,\tau)dx} + a^2\int_0^T\int_0^l u_x^2(x,t)dx dt$$

$$\leq F(\tau) + \Omega(\tau) \leq (M+1)F(\tau).$$

$$= \tilde{M}\left(\int_0^l \varphi^2 dx + \int_0^T\int_0^l f^2 dx dt\right).$$

$$\sup_{0\leq t\leq T}\int_0^l u^2(x,t)dx + 2a^2\int_0^T\int_0^l u_x^2(x,t)dx dt$$

$$\leq \tilde{M}\left(\int_0^l \varphi^2 dx + \int_0^T\int_0^l f^2 dx dt\right)$$

Remark: 若 $\varphi=0$, $f=0$. 则.

$$\sup_{0\leq t\leq T}\int_0^l u^2(x,t)dx = 0. \qquad \int_0^T\int_0^l u_x^2(x,t)dx dt = 0$$

$$\Longrightarrow \qquad u=0 \quad \forall (x,t)\in Q_T$$

P158: 得到 $\frac{\partial u}{\partial t}$ 的一个上界估计.

$u_t - a^2 u_{xx} = f$ 两侧乘 $\frac{\partial u}{\partial t}$ 得到的

位势方程:

$$\begin{cases} -\Delta u + c(x)u = f(x) & x\in\Omega. \\ u|_{\partial\Omega} = 0. \end{cases}$$

$c(x) \geq c_0 > 0$, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ 为(*)的解，则有估计

$$\boxed{\int_\Omega |\nabla u|^2 dx + \frac{c_0}{2}\int_\Omega |u(x)|^2 dx \leq M \int_\Omega |f|^2 dx.}$$

√.

Proof: 在 $-\Delta u + c(x)u = f$ 两侧乘 u 积分

$$\int_\Omega -u\Delta u + c(x)u^2 \, dx = \int_\Omega fu \, dx$$

$$\int_\Omega u\Delta u \, dx = -\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} u \frac{\partial u}{\partial n} dS$$

$$= -\int_\Omega |\nabla u|^2 dx.$$

$$\int_\Omega -u\Delta u + c(x)u^2 dx$$

$$= \int_\Omega |\nabla u|^2 + c(x)u^2 dx \geq \int_\Omega |\nabla u|^2 dx + c_0\int_\Omega u^2 dx.$$

$$\Rightarrow \int_\Omega |\nabla u|^2 dx + c_0\int_\Omega u^2 dx \leq \int_\Omega fu \, dx.$$

$$fu = \varepsilon u \cdot \frac{1}{\varepsilon} f \qquad (\varepsilon > 0)$$

$$\leq \frac{1}{2}\varepsilon^2 u^2 + \frac{1}{2\varepsilon^2} f^2.$$

$$\Rightarrow \int_\Omega |\nabla u|^2 dx + \underline{c_0\int_\Omega u^2 dx} \leq \int_\Omega fu \, dx$$

$$\leq \underline{\frac{1}{2}\varepsilon^2\int_\Omega u^2 dx} + \frac{1}{2\varepsilon^2}\int_\Omega f^2 dx$$

$$\Rightarrow \int_\Omega |\nabla u|^2 dx + \left(c_0 - \frac{1}{2}\varepsilon^2\right)\int_\Omega u^2 dx \leq \frac{1}{2\varepsilon^2}\int_\Omega f^2 dx.$$

取 ε 满足 $c_0 - \frac{1}{2}\varepsilon^2 = \frac{c_0}{2} \Longleftrightarrow \varepsilon^2 = c_0.$

$$\Rightarrow \int_\Omega |\nabla u|^2 dx + \frac{c_0}{2}\int_\Omega u^2 dx \leq \frac{1}{2c_0}\int_\Omega f^2 dx.$$

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases} \quad ☆$$

Lemma (Poincaré Inequality) 设 Ω, $u\in C_0^1(\Omega)$，则

$$\boxed{\|u\|_{L^2(\Omega)} \leqslant 2d\,\|\nabla u\|_{L^2(\Omega)}}$$

$d = \operatorname{diam}(\Omega) = \sup_{x,y\in\Omega} |x-y|$.

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}.$$

对(☆)两侧乘 u 积分

$$\int_\Omega -\Delta u\, u\, dx = \int_\Omega f u\, dx \leqslant \frac{1}{2}\varepsilon^2 \int u^2 dx + \frac{1}{2\varepsilon^2}\int_\Omega f^2 dx$$

$$\overset{\|}{\int_\Omega |\nabla u|^2 dx.}$$

$$\int_\Omega u^2 dx \leqslant 4d^2 \int_\Omega |\nabla u|^2 dx$$

$$\int_\Omega |\nabla u|^2 dx \leqslant \frac{1}{2}\varepsilon^2\cdot 4d^2 \int_\Omega |\nabla u|^2 dx + \frac{1}{2\varepsilon^2}\int_\Omega f^2 dx.$$

取 $\frac{1}{2}\varepsilon^2\cdot 4d^2 = \frac{1}{2}$，则 $\varepsilon = \frac{1}{2d}$

$$\Rightarrow \quad \boxed{\int_\Omega |\nabla u|^2 dx \leqslant 4d^2 \int_\Omega f^2 dx}.$$

$$c\int_\Omega |u|^2 dx \leqslant$$

Remark: $f = 0 \Rightarrow u = 0$.

阅读 附注3 (习题13).

Fourier 变换:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\, e^{-i\xi x}\, dx.$$

反演公式:

$f \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$，则有

$$f(x) = \lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} \hat{f}(\lambda)\, e^{i\lambda x}\, d\lambda$$

$\mathcal{F} : f \longmapsto \hat{f}$

$\mathcal{F}^{-1} : \hat{f} \longmapsto (\hat{f})^{\vee}$

Fourier 变换性质.

① 线性性 $\quad (\alpha f + \beta g)^{\wedge} = \alpha \hat{f} + \beta \hat{g}$

② 微商性：

$$\left(\frac{df}{dx}\right)^{\wedge} = i\lambda \hat{f}.$$

Proof:

$$\left(\frac{df}{dx}\right)^{\wedge} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{df}{dx}(x)\, e^{-i\lambda x}\, dx = \frac{-1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, \frac{d}{dx} e^{-i\lambda x}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}}\, i\lambda \int_{\mathbb{R}} f(x)\, e^{-i\lambda x}\, dx$$

$$= i\lambda \hat{f}$$

③ 卷积性质

$f, g \in L^1(\mathbb{R})$，则

$$(f * g)^{\wedge} = \sqrt{2\pi}\, \hat{f} \hat{g}$$

其中 $f * g(x) = \int_{\mathbb{R}} f(x-t)\, g(t)\, dt.$

Proof:

$$(f*g)^{\wedge}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f*g(x)\, e^{-i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-t)\, g(t)\, dt \right) e^{-i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-t)\, g(t)\, dt \right) e^{-i\lambda (x-t)} \cdot e^{-i\lambda t}\, dx.$$

$$= \int_{\mathbb{R}} g(t) \underline{\left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-t)\, e^{-i\lambda (x-t)}\, dx \right)} e^{-i\lambda t}\, dt.$$

$$= \int_{\mathbb{R}} g(t)\, \hat{f}(\lambda)\, e^{-i\lambda t}\, dt.$$

$= \sqrt{2\pi}\, \hat{f}(\lambda)\, \hat{g}(\lambda)$

④伸缩性质

$f \in L^1(\mathbb{R}^n)$. 定义：☆.

$$f_t(x) = \frac{1}{t} f\left(\frac{x}{t}\right). \quad (t>0)$$

$$\hat{f_t}(\lambda) = \hat{f}(\lambda t)$$

Proof: $\hat{f_t}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{t} f\left(\frac{x}{t}\right) e^{-i\lambda x} dx$ $\quad \frac{x}{t} \mapsto y$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y)\, e^{-i\lambda t y}\, dx.$$

$$= \hat{f}(\lambda t)$$

例：

$f(x) = e^{-\frac{1}{2}x^2}$. $\quad \hat{f}(\lambda) = e^{-\frac{1}{2}\lambda^2}$. Fourier变换不动点

Proof: $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} e^{-i\lambda x} dx.$

$$\frac{d}{d\lambda} \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \frac{d}{d\lambda} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} e^{-i\lambda x} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} \cdot (-ix) e^{-i\lambda x} dx.$$

$$= -i \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{-\frac{1}{2}x^2} e^{-i\lambda x} dx$$

$$= i \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} e^{-i\lambda x}\, d\frac{1}{2}x^2.$$

$$= i \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda x}\, d e^{-\frac{1}{2}x^2}$$

$$= -\lambda \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}x^2} e^{-i\lambda x} dx$$

$$= -\lambda \hat{f}(\lambda).$$

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} dx$$

ODE: $\begin{cases} \frac{d}{d\lambda}\hat{f}(\lambda) + \lambda \hat{f}(\lambda) = 0 \\ \hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} dx. = 1 \end{cases}$ $\quad$ Gauss 积分

$g' + xg = 0 \qquad \frac{d}{dx} g = -xg$

$$\frac{dg}{g} = -x\,dx$$

$$d\log g = d\left(-\frac{1}{2}x^2\right)$$

$$g = C\exp\left(-\frac{1}{2}x^2\right)$$

$$\Rightarrow \hat{f}(\lambda) = \underline{C}\exp\left(-\frac{1}{2}\lambda^2\right) = \exp\left(-\frac{1}{2}\lambda^2\right)$$

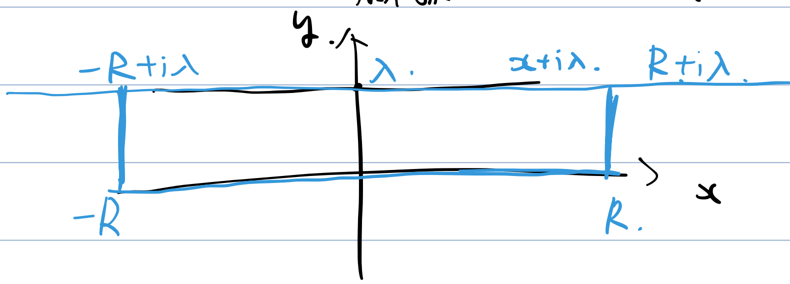
$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}x^2} e^{-i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}x^2 - i\lambda x}\,dx$$

$$\frac{1}{2}x^2 + i\lambda x = \frac{1}{2}(x+i\lambda)^2 + \frac{\lambda^2}{2}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}(x+i\lambda)^2}\cdot e^{-\frac{\lambda^2}{2}}\,dx$$

$$= e^{-\frac{\lambda^2}{2}}\cdot \underbrace{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}(x+i\lambda)^2}\,dx}_{=1}$$



例: $f(x) = e^{-Ax^2}$. $\qquad \frac{1}{2}x^2 \xrightarrow{x \mapsto \sqrt{\frac{A}{2}}\,y} Ay^2$

$\hat{f} = ?$

$g(x) = e^{-\frac{1}{2}x^2}$. $\qquad \hat{g}(\lambda) = e^{-\frac{1}{2}\lambda^2}$.

$\sqrt{2A}\,f(x) = \sqrt{2A}\,g(\sqrt{2A}\,x) \qquad t = \sqrt{\frac{2}{A}}$

$$\| $$

$$g_{\frac{1}{\sqrt{2A}}}(x)$$

则 $\left(g_{\sqrt{\frac{2}{A}}}\right)^{\wedge}(\lambda) = \hat{g}\left(\sqrt{\frac{1}{2A}}\,\lambda\right) = \exp\left(-\frac{1}{4A}\lambda^2\right)$

$$\Downarrow$$

$$\Rightarrow \sqrt{2A}\,\hat{f}(\lambda) = \exp\left(-\frac{1}{4A}\lambda^2\right)$$

$$\Rightarrow \hat{f}(\lambda) = \frac{1}{\sqrt{2A}} \exp\left(-\frac{1}{4A}\lambda^2\right) \quad ☆$$

热方程：

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t) \\ u(x,0) = \varphi(x). \end{cases} \quad ☆$$

对 (☆) 关于 x 作 Fourier 变换 (t 看作常数)

$$\hat{u_t}(\lambda,t) = \frac{d}{dt}\hat{u}(\lambda,t)$$

$$\hat{u_{xx}}(\lambda,t). = (i\lambda)^2\hat{u}(\lambda,t) = -\lambda^2\hat{u}(\lambda,t).$$

$$\hat{f}(\lambda,t). \qquad \hat{\varphi}(\lambda).$$

则 (☆) 变为：

$$\begin{cases} \frac{d}{dt}\hat{u}(\lambda,t) + a^2\lambda^2\hat{u}(\lambda,t) = \hat{f}(\lambda,t) \\ \hat{u}(\lambda,0) = \hat{\varphi}(\lambda). \end{cases}$$ 关于 t 的 ODE.

$$\frac{d}{dt}\left(\hat{u}(\lambda,t)e^{a^2\lambda^2 t}\right) = \hat{f}(\lambda,t)e^{a^2\lambda^2 t}$$

$$\hat{u}(\lambda,t)e^{a^2\lambda^2 t} - \hat{\varphi}(\lambda) = \int_0^t \hat{f}(\lambda,\tau)e^{a^2\lambda^2\tau}\,d\tau.$$

$$\Rightarrow \hat{u}(\lambda,t) = e^{-a^2\lambda^2 t}\hat{\varphi}(\lambda) + \int_0^t \hat{f}(\lambda,\tau)e^{-a^2\lambda^2(t-\tau)}\,d\tau.$$

$$\Downarrow \mathcal{F}^{-1}$$

$$u(x,t).$$

$$\sqrt{4\pi}\,e^{-a^2\lambda^2(t-\tau)} \stackrel{\Delta}{=} \hat{g}_{t-\tau}(\lambda)$$

令 $\hat{g}_t(\lambda) = \frac{1}{\sqrt{4\pi}}e^{-a^2\lambda^2 t}$

$$\underbrace{\sqrt{2\pi}\,\hat{g}_t(\lambda)\cdot\hat{\varphi}(\lambda)}_{\| \atop (g_t*\varphi)^{\wedge}(\lambda).} = e^{-a^2\lambda^2} + \hat{\varphi}(\lambda)$$

$$\Rightarrow \hat{u}(\lambda,t) = \sqrt{2\pi}\,\hat{g}_t(\lambda)\cdot\hat{\varphi}(\lambda) + \int_0^t \hat{f}(\lambda,\tau)\,\hat{g}_{t-\tau}(\lambda)\sqrt{2\pi}\,d\tau.$$

$$f(x,t) =: f_t(x)$$

$$\hat{f}(\lambda,t) = \hat{f}_t(\lambda).$$

$$= \sqrt{2\pi}\,\hat{g}_t(\lambda)\hat{\varphi}(\lambda) + \int_0^t \hat{f}_\tau(\lambda)\,\hat{g}_{t-\tau}(\lambda)\sqrt{2\pi}\,d\tau.$$

$$= (g_t*\varphi)^{\wedge}(\lambda) + \int_0^t (f_\tau * g_{t-\tau})^{\wedge}(\lambda)\,d\tau.$$

$$\Rightarrow u(x,t) = g_t*\varphi(x) + \int_0^t f_\tau * g_{t-\tau}(x)\,d\tau.$$

$$= \int_{\mathbb{R}^n} g_t(x-y)\,\varphi(y)\,dy$$

$$+ \int_0^t \left(\int_{\mathbb{R}} f_\tau(x-y)\,g_{t-\tau}(y)\,dy\right) d\tau$$

$$= \int_{\mathbb{R}^n} g_t(x-y)\,\varphi(y)\,dy + \int_0^t \left(\int_{\mathbb{R}} f(x-y,\tau)\,g_{t-\tau}(y)\,dy\right) d\tau.$$

$$g_t(x) \longrightarrow \delta(x) \qquad (t\to 0).$$