分离变量法.

(Strum-Liouville 问题) (P67)

在 $[0,l]$ 上的特征值问题

$$(☆)\begin{cases} X''+\lambda X=0 & 0<x<l. \quad \text{ODE} \\ -\alpha_1 X'(0)+\beta_1 X(0)=0. \\ \alpha_2 X'(l)+\beta_2 X(l)=0. \end{cases} \Rightarrow \text{边界条件.}$$

其中 $\alpha_i \geq 0$, $\beta_i \geq 0$, $\alpha_i+\beta_i \neq 0$, 则有如下结论.

① 所有特征值均非负. 特别的 $\beta_1+\beta_2>0$, 则特征值都是正数

$\{X_n\}_{n=1}^{\infty}$ $\quad X_n''+\lambda_n X_n=0.$

②. 特征值是单调增且趋于 $+\infty$.

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \quad \lim_{n\to\infty}\lambda_n=+\infty.$$

③. $\forall\, i,j$, $i\neq j$, X_i (λ_i 对应的) 与 X_j (λ_j 对应) 正交, 即,

$$\int_0^l X_i(x)\,X_j(x)\,dx=0.$$

④. $\{X_i\}$ 构成 $L^2[0,l]$. 完备正交系, $\forall f\in L^2[0,l]$,

$$\int_0^l |f(x)|^2dx<\infty.$$

则存在 $\{C_n\}_{n=1}^{\infty}\subset\mathbb{R}$ 使得

$$f(x)=\sum_{n=1}^{\infty}\boxed{C_n}X_n(x). \quad L^2 \text{ 收敛}$$

即. $\lim_{N\to\infty}\int_0^l\left|f(x)-\sum_{n=1}^{N}C_nX_n(x)\right|^2dx=0.$

Remark: $L^2[0,l]$ 上定义内积.

$$(f,g)=\int_0^l f(x)g(x)\,dx. \quad f,g\in L^2[0,l].$$

③ 说的是: $\int_0^l X_i(x)\,X_j(x)\,dx=0 \quad i\neq j$

$$(X_i, X_j) = 0 \quad \text{if } i \neq j$$

④ 证明是

$$f = \sum C_i X_i$$

两侧同时对 X_n 作内积有:

$i \neq n$, 则 $(X_i, X_n) = 0$

$$(f, X_n) = (\sum C_i X_i, X_n).$$

$$= \sum C_i (X_i, X_n) = C_n (X_n, X_n).$$

$$\Rightarrow \quad C_n = \frac{(f, X_n)}{(X_n, X_n)}$$

$$= \frac{\int_0^l f(x) X_n(x) dx}{\int_0^l (X_n(x))^2 dx}$$

综上: $$f(x) = \sum_{n=1}^{\infty} \left(\frac{\int_0^l f(x) X_n(x) dx}{\int_0^l (X_n(x))^2 dx} \right) X_n(x).$$

例: $$\begin{cases} X'' + \lambda X = 0. & 0 < x < l. \\ X(0) = 0 & ① \\ X(l) = 0 & ② \end{cases}$$

Solution: 由① $\Rightarrow$ $\lambda \geq 0$.

$$X'' + \lambda X = 0$$

的解为 $X(x) = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x).$

代入边界条件①

$$X(0) = C_2 = 0.$$

$$\Rightarrow \quad X(x) = C_1 \sin(\sqrt{\lambda} x)$$

代入边界条件②

$$X(l) = C_1 \sin(\sqrt{\lambda}\, l) = 0.$$

$\sin x = 0 \Leftrightarrow x = k\pi \ (k \in \mathbb{Z}).$

$\sqrt{\lambda}\, l = n\pi$ ① λ 非负

$\Leftrightarrow \boxed{\lambda = \left(\frac{n\pi}{l}\right)^2 \quad n \in \mathbb{Z}_{>0}}.$ ② $\lambda_n \to +\infty$

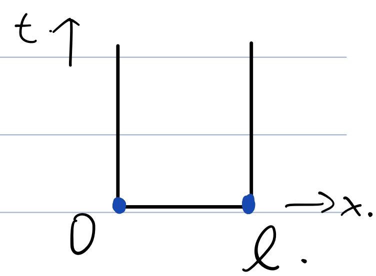
令 $\lambda_n = \left(\frac{n\pi}{l}\right)^2$，则

$$X_n(x) = \sin\left(\frac{n\pi}{l} x\right)$$

即 $\left\{\left(\left(\frac{n\pi}{l}\right)^2, \sin\left(\frac{n\pi}{l} x\right)\right)\right\}_{n=1}^{\infty}$ 为☆的解.

波动方程混合问题(分离变量) Case 1.

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \ ☆ \ \checkmark & (x,t) \in (0,l) \times (0,+\infty). \\ \boxed{u(0,t) = u(l,t) = 0 \quad t > 0} \\ u(x,0) = \varphi(x) & 0 \le x \le l \ ☆ \\ u_t(x,0) = \psi(x) & 0 \le x \le l. \ ☆ \end{cases}$$



$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t).$$

想法：使 X_n 满 $X_n'' + \lambda_n X_n = 0 \longrightarrow$ 必定可解

$$u_{tt} = \sum_{n=1}^{\infty} X_n(x) T_n''(t).$$

$$u_{xx} = \sum_{n=1}^{\infty} X_n''(x) T_n(t). = -\sum_{n=1}^{\infty} \lambda X_n(x) T_n(t).$$

$$u_{tt} - a^2 u_{xx} = \sum_{n=1}^{\infty} X_n(x) \left(T_n''(t) + \lambda a^2 T_n(t) \right) = 0$$

$$\downarrow$$

$$T_n'' + \lambda a^2 T_n = 0.$$

想法.

$u(0,t) = \sum X_n(0) T_n(t) = 0.$ 随t变动. $X_n(0) = 0$

$u(\ell,t) = \sum X_n(\ell) T_n(t) = 0$ $\quad\Rightarrow\quad X_n(\ell)=0.$

得到 $\begin{cases} X_n'' + \lambda_n X = 0 & 0<x<\ell. \\ X_n(0) = X_n(\ell) = 0 \end{cases}$

一组解 $\{X_n\}$ $\{\lambda_n\}$.

$u(x,0) = \sum X_n(x) T_n(0) = \varphi(x) = \sum \varphi_n X_n(x).$

$\hookrightarrow T_n(0) = \varphi_n.$

$u_t(x,0) = \sum X_n(x) T_n'(0) = \psi(x) = \sum \psi_n X_n(x).$

$\hookrightarrow T_n'(0) = \psi_n.$

$\begin{cases} T_n''(t) + a^2 \lambda_n T_n = 0. \\ T_n(0) = \varphi_n, \quad T_n'(0) = \psi_n. \end{cases}$

步骤:

$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \\ u(0,t) = u(\ell,t) = 0. \\ u(x,0) = \varphi(x) \quad u_t(x,0) = \psi(x). \end{cases}$ $\qquad u = \sum X_n(x) T_n(t).$

①. $\begin{cases} X_n'' + \lambda_n X_n = 0 \\ X_n(0) = X_n(\ell) = 0. \end{cases}$

$X_n(x) = \sin\left(\frac{n\pi}{\ell}x\right). \qquad n \geq 1 \qquad \lambda_n$ 可不用算.

②. 代入方程找 T_n 满足的 ODE.

$u_{xx} = \sum T_n(t) X_n''(x) = -\sum \left(\frac{n\pi}{\ell}\right)^2 T_n(t) \sin\left(\frac{n\pi}{\ell}x\right)$

$u_{tt} = \sum T_n''(t) \sin\left(\frac{n\pi}{\ell}x\right).$

$0 = u_{tt} - a^2 u_{xx} = \sum \sin\left(\frac{n\pi}{\ell}x\right)\left(\boxed{T_n''(t) + \left(\frac{n\pi a}{\ell}\right)^2 T_n(t)}\right)$

$$u(x,0)=\sum T_n(0)\sin\left(\frac{n\pi}{l}x\right)=\varphi(x)=\sum \varphi_n \sin\left(\frac{n\pi}{l}x\right)$$

$$u_t(x,0)=\sum T_n'(0)\sin\left(\frac{n\pi}{l}x\right)=\psi(x)=\sum \psi_n \sin\left(\frac{n\pi}{l}x\right).$$

$$\varphi_n=\frac{2}{l}\int_0^l \varphi(x)\sin\left(\frac{n\pi}{l}x\right)dx.$$

$\hookrightarrow$ 周期.

$$\psi_n=\frac{2}{l}\int_0^l \varphi(x)\sin\left(\frac{n\pi}{l}x\right)dx.$$

$$\begin{cases} T_n''(t)+\left(\frac{n\pi a}{l}\right)^2 T_n(t)=0 \quad ☆ \\ T_n(0)=\varphi_n \qquad T_n'(0)=\psi_n. \end{cases}$$

$$T_n(t)=C_1\sin\left(\frac{n\pi a}{l}t\right)+C_2\cos\left(\frac{n\pi a}{l}t\right).$$

$$u(x,t)=\sum T_n(t)\sin\left(\frac{n\pi}{l}x\right).$$

$$\begin{cases} u_{tt}-a^2u_{xx}=f(x,t). \quad ☆ \\ u(0,t)=u(l,t)=0. \\ u(x,0)=\varphi(x) \qquad u_t(x,0)=\psi(x). \end{cases}$$

Case 2.

$$u(x,t)=\sum X_n(t)\,T_n(t).$$

步骤① 完全一样

步骤②:

$$u_{tt}-a^2u_{xx}=\sum X_n(t)\left(T_n''(t)+\lambda a^2 T_n(t)\right)$$

$$=f(x,t).=\sum f_n(t)\,X_n(t)$$

$$f_n(t)=\frac{\int_0^l f(x,t)\,X_n(x)\,dx}{\int_0^l |X_n(x)|^2dx}.$$

$$\left.\begin{cases} T_n''+\lambda a^2T_n=f_n(t). \\ T_n(0)=\varphi_n,\ T_n'(0)=\psi_n. \end{cases}\right\} \xrightarrow{\text{解 ODE.}} \text{得到答案.}$$

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0. \\ u(0,t) = g_1(t), \quad u(l,t) = g_2(t) \\ u(x,0) = \varphi(x) \quad u_t(x,0) = \psi(x). \end{cases}$$

Case 3

想法：作变量替换 $u \mapsto v$，使 $v(0,t) = v(l,t) = 0$

令 $v(x,t) = u(x,t) - \boxed{\frac{l-x}{l} g_1(t) - \frac{x}{l} g_2(t)}$.

$$\begin{cases} v(0,t) = u(0,t) - g_1(t) = 0 \\ v(l,t) = u(l,t) - g_2(t) = 0. \end{cases}$$

则 $v(x,0) = u(x,0) - \frac{l-x}{l} g_1(0) - \frac{x}{l} g_2(0)$.

$= \varphi(x) - \frac{l-x}{l} g_1(0) - \frac{x}{l} g_2(0) =: \tilde{\varphi}(x)$.

$v_t(x,0) = u_t(x,0) - \frac{l-x}{l} g_1'(0) - \frac{x}{l} g_2'(0)$

$= \psi(x) - \frac{l-x}{l} g_1'(0) - \frac{x}{l} g_2'(0) =: \tilde{\psi}(x)$.

$v_{tt} - a^2 v_{xx} = \underbrace{u_{tt} - a^2 u_{xx}}_{0} - \underline{\left(\frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2}\right)\left(\frac{l-x}{l} g_1(t) - \frac{x}{l} g_2(t)\right)}$.

$=: f(x,t)$.

$\Rightarrow$ v 满足形如 Case 2 的方程

热方程：

$$\begin{cases} u_t - a^2 u_{xx} = \boxed{f(x,t)} & 0<x<l, \quad 0<t<T \\ u(x,0) = \varphi(x) & 0 \le x \le l. \\ \boxed{u(0,t) = u(l,t) = 0.} & 0 \le t \le T. \end{cases}$$

$u = \sum X_n(x) T_n(t)$.

① 求解 $\begin{cases} X_n'' + \lambda X_n = 0. \end{cases}$

$$X(0) = X(l) = 0.$$

② $u_t = \sum X_n(x) T_n'(t)$

$u_{xx} = \sum X_n''(x) T_n(t) = -\sum \lambda X_n(x) T_n(t)$

$\Rightarrow u_t - a^2 u_{xx} = \sum (T_n'(t) + a^2 \lambda T_n(t)) X_n(x).$

$= f(x,t) = \sum f_n(t) X_n(t).$

$\Rightarrow$ ① $T_n'(t) + a^2 \lambda T_n(t) = f_n(t).$

$u(x,0) = \sum X(x) T(0) = \varphi(x) = \sum \varphi_n X_n(t).$

② $T_n(0) = \varphi_n.$

①② $\Rightarrow$ T_n 解得 $(\cdots)$ — P163, 8 ~ 10.

$u(x,t) = \sum T_n(t) X_n(x)$ — P164, 22, ~ 26

能量估计 (Gronwell inequality + 分部积分).

THM (Gronwell inequality) P87

$G(\tau) : [0,T] \longrightarrow \mathbb{R}$ 非负，C^1 函数，$G(0)=0.$

$G(\tau) = \int_0^\tau f(x)\,dx.$ $G(0)=0.$

且 $\forall \tau \in [0,T]$ 有

$$\frac{dG(\tau)}{d\tau} \leq C G(\tau) + F(\tau)$$

其中 $C > 0$ 常数，$F(\tau)$ 不减非负可积函数，则.

$$\frac{dG}{d\tau} \leqslant e^{C\tau} F(\tau).$$

$$G(\tau) \leqslant C^{-1} (e^{C\tau} - 1) F(\tau).$$

Proof:

$$\frac{dG}{dt} \le C\,G(T) + F(T).$$

启发式题: $f' = Cf + g(x)$ “积分因子”

$$\underbrace{e^{-Cx}f' - Ce^{-Cx}f} = ge^{-Cx}.$$

$$\frac{d}{dx}\left(fe^{-Cx}\right) = ge^{-Cx}.$$

$F(T)$ 非减

$$\int_0^t \frac{d}{d\tau}\left(Ge^{-C\tau}\right)d\tau \le \int_0^t F(\tau)\, e^{-C\tau}\, d\tau.$$

$$\le F(t)\int_0^t e^{-C\tau}\,d\tau$$

$$G(t)e^{-Ct} \qquad =: M\cdot F(t)$$

$\tau\in[0,T]$

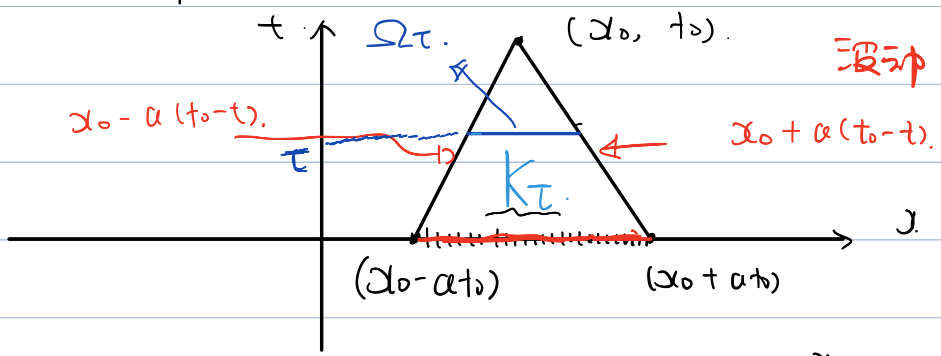
$$\Rightarrow \quad G(t) \le Me^{CT}F(t)$$

又 $\frac{dG}{dt} \le CG + F \le \tilde{M}F(t).$

能量不等式.

设 $u\in C^1(\bar{\Omega})\cap C^2(\Omega)$ Ω 为上半平面. $\mathbb{R}\times(0,\infty)$

$$\begin{cases} u_{tt} - a^2u_{xx} = f(x,t) \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$



$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{2a}\int_{x-at}^{x+at}\varphi(\xi)\,d\xi\right) + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,d\xi$$

$$+\int_0^t \left(\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\,d\xi\right)d\tau$$

Pf:

$$\boxed{E(t)=\frac{1}{2}\int_{x_0-a(t_0-t)}^{x_0+a(t_0-t)} \underbrace{u_t^2}_{\text{动能}}+\underbrace{a^2u_x^2}_{\text{势能}}\,dx.}$$ 波动方程 → 能量.

动能+势能

$$=\frac{1}{2}\int_{\Omega_t} u_t^2+a^2u_x^2\,dx.$$

$\boxed{E(t)\geq 0}$

[变上限积分求导: $\frac{d}{dx}\int_0^{g(x)} f(x,y)\,dy$

$$=g'(x)f(x,g(x))+\int_0^{g(x)}\frac{\partial}{\partial x}f(x,y)\,dy.$$

$$\frac{dE(t)}{dt}=-\frac{1}{2}a\left(u_t^2+a^2u_x^2\right)\Big|_{x-a(t_0-t)}^{x+a(t_0-t)}$$

$$+\frac{1}{2}\int_{\Omega_t}\frac{\partial}{\partial t}\left(u_t^2+a^2u_x^2\right)dx.$$

$$=\int_{\Omega_t}\left(u_tu_{tt}+a^2u_xu_{xt}\right)dx$$ u_t.

$$-\frac{1}{2}a\left(u_t^2+a^2u_x^2\right)\Big|_{x-a(t_0-t)}^{x+a(t_0-t)}$$

分部积分.

$$\int_{\Omega_t}u_xu_{xt}\,dx=\int_{\Omega_t}u_x\,du_t=u_xu_t\Big|_{x-a(t_0-t)}^{x+a(t_0-t)}$$

$$-\int_{\Omega_t}u_tu_{xx}\,dx$$

$$=\int_{\Omega_t}u_t(u_{tt}-a^2u_{xx})\,dx$$

$$+\underbrace{\left(a^2u_xu_t-\frac{1}{2}(au_t^2+a^3u_x^2)\right)}_{\leq 0}\Big|_{x-a(t_0-t)}^{x+a(t_0-t)}.$$

$$a^2u_xu_t=\sqrt{a}\,u_t\cdot\sqrt{a^3}\,u_x\leq\frac{1}{2}\left(au_t^2+a^3u_x^2\right)$$

$$\leq\int_{\Omega_t}u_tf(x,t)\,dx$$

$$\Rightarrow \quad \frac{dE}{dt} \le \int_{\Omega_t} u_t f(x,t)\,dx$$

从 0 到 τ 积分

$$E(\tau) - E(0) \le \int_0^\tau \int_{\Omega_t} u_t f(x,t)\,dx\,dt.$$

$$\le \int_{K_\tau} u_t f(x,t)\,dx\,dt$$

$$\le \frac{1}{2}\int_{K_\tau} u_t^2\,dx\,dt + \frac{1}{2}\int_{K_\tau} f^2(x,t)\,dx\,dt.$$

注意: $E(T) = \frac{1}{2}\int_{\Omega_t} u_t^2 + a^2 u_x^2\,dx$

$\int_0^\tau E(t)\,dt = \frac{1}{2}\int_{K_\tau} u_t^2 + a^2 u_x^2\,dx =: \Omega(\tau)$

$$\Rightarrow \quad \frac{d\Omega}{d\tau} \le E(0) + \frac{1}{2}\int_{K_\tau} f^2(x,t)\,dx\,dt + \frac{1}{2}\int_{K_\tau} u_t^2\,dx\,dt$$

$$\le \underbrace{E(0) + \frac{1}{2}\int_{K_\tau} f^2(x,t)\,dx\,dt}_{4} + \frac{1}{2}\Omega(\tau).$$

由. Gronwell inequality:

$$\Omega(t) \le M\left(E(0) + \int_{K_\tau} f^2(x,t)\,dx\,dt.\right)$$

$$\Downarrow$$

$$\int_{\Omega_0} u_t^2(x,0) + a^2 u_x^2(x,0)\,dx$$

$$= \int_{\Omega_0} \psi(x)^2 + a^2 \cdot (\varphi'(x))^2\,dx.$$

即.

$E(t) = 0.$

$$\int_{K_\tau} u_t^2 + a^2 u_x^2\,dx\,dt \le M\left(\int_{\Omega_0} \psi(x)^2 + a^2(\varphi'(x))^2\,dx + \int_{K_\tau} f^2(x,t)\,dx\,dt\right).$$

u的能量.

若: $\varphi = \psi = f = 0$. 则 $u_t^2 + a^2 u_x^2 = 0$

$$\Rightarrow \quad u_t,\ u_x = 0.$$

"唯一性"：u, v 满足

$$☆☆\quad \begin{cases} u_{tt}-a^2u_{xx}=f\\ u(x,0)=\varphi\\ u_t(x,0)=\psi\end{cases}$$

则 $u-v$ 满足

$$☆\quad \left.\begin{cases} u_{tt}-a^2u_{xx}=0\\ u(x,0)=0\\ u_t(x,0)=0\end{cases}\right\} \Longrightarrow u\equiv 0$$

$u-v$ 满足☆，但☆的解只有零解，则 $u-v\equiv 0$ 即.

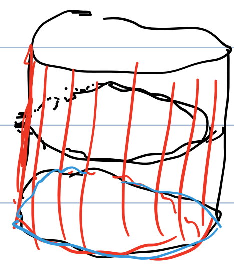
☆☆ 解唯一.

例题

$$\begin{cases} u_t-a^2\Delta u=0.\\ \left.\left(\alpha\dfrac{\partial u}{\partial n}+\sigma u\right)\right|_{\Sigma}=0. \quad ☆\\ u|_{t=0}=\varphi\end{cases}$$

$G=\Omega\times(0,\infty)$

$\Sigma=\partial\Omega\times[0,\infty)$



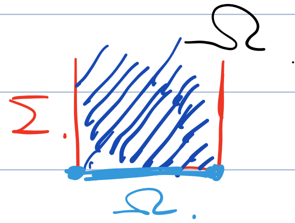
$$E(t)=\frac{1}{2}\int_\Omega u^2(x,y,z,t)\,dxdydz$$

证明：① $\dfrac{dE}{dt}\le 0$

② 解唯一

① $\dfrac{dE}{dt}=\int_\Omega u_t u\,dxdydz.$

$\Big\|\ u_t=a^2\Delta u.$



$$=\int_\Omega a^2 u\Delta u\,dxdydz$$

$$= a^2 \int_\Omega u \Delta u \, dx\,dy\,dz.$$

$$= a^2 \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, dS - a^2 \int_\Omega |\nabla u|^2 \, dx\,dy\,dz$$

$\partial\Omega \subset \Sigma$

$$= -a^2 \int_{\partial\Omega} u \frac{\sigma}{\alpha} u \, dS - a^2 \int_\Omega |\nabla u|^2 \, dx\,dy\,dz$$

$$= -a^2 \frac{\sigma}{\alpha} \int_{\partial\Omega} u^2 \, dS - a^2 \int_\Omega |\nabla u|^2 \, dx\,dy\,dz.$$

$$\leq 0.$$

②. 当 $\varphi = 0$ 时
$$\begin{cases} u_t - a^2 \Delta u = 0 \\ \alpha \frac{\partial u}{\partial n} + \sigma u \big|_\Sigma = 0 \\ u\big|_{t=0} = 0. \end{cases}$$

去证明: u 只有零解. 即 $E(t) = \frac{1}{2}\int u^2 \, dx\,dy\,dz = 0$

由①: $\frac{dE}{dt} \leq 0.$

$$\Rightarrow \quad 0 \geq \int_0^T \frac{dE}{dt} dt = E(T) - E(0)$$

$$\Rightarrow \quad E(T) \leq E(0) = \frac{1}{2}\int u(x,y,z,0)^2 \, dx\,dy\,dz = \frac{1}{2}\int 0 \, dx\,dy\,dz = 0$$

$\Rightarrow$ $\forall T>0$ 有 $0 \geq E(T) \geq 0$

即 $E(T) = 0 \quad \forall \ T>0$

故 $u = 0 \quad \forall \ x,y,z,t$ i.e. ☆ 只有零解.

半无界