Green.

① $\int_\Omega u\Delta v - v\Delta u\,dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,dS(x)$

②. $\Gamma(x;\xi) = \begin{cases} \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x-\xi|^{n-2}} & n\geq 3 \\ \frac{1}{2\pi}\log\frac{1}{|x-\xi|} & n=2 \end{cases}$

③. $u(\xi) = -\int_\Omega \Gamma(x;\xi)\Delta u\,dx + \int_{\partial\Omega}\left(\Gamma(x;\xi)\frac{\partial u}{\partial n} - \frac{\partial\Gamma(x;\xi)}{\partial n}u\right)dS.$

④. $0 = -\int_\Omega g(x;\xi)\Delta u\,dx + \int_{\partial\Omega}\left(g(x;\xi)\frac{\partial u}{\partial n} - \frac{\partial g}{\partial n}(x;\xi)u\right)dS.$

$G(x;\xi) = g(x;\xi) + \Gamma(x;\xi)$

$$u(\xi) = -\int_\Omega G(x;\xi)\Delta u\,dx + \int_{\partial\Omega} G(x;\xi)\boxed{\frac{\partial u}{\partial n}} - \frac{\partial G}{\partial n}(x;\xi)u)\,dS.$$

使之消失

$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$

$G: \begin{cases} -\Delta G = \delta(x-\xi) & \Omega \\ G = 0 & \partial\Omega \end{cases}$ $\quad g: \begin{cases} -\Delta g(x;\xi) = 0 & \Omega \\ g = -\Gamma & \partial\Omega. \end{cases}$

求 Green function.

① 球 $B(0,R)$ 上

$G(x;\xi) = \Gamma(x;\xi) - \Phi\left(\frac{|\xi|}{R}(x-\tilde{\xi})\right)$ $\quad \Phi(x) = \Gamma(x;0)$

$= \Phi(x-\xi) - \Phi\left(\frac{|\xi|}{R}(x-\tilde{\xi})\right).$

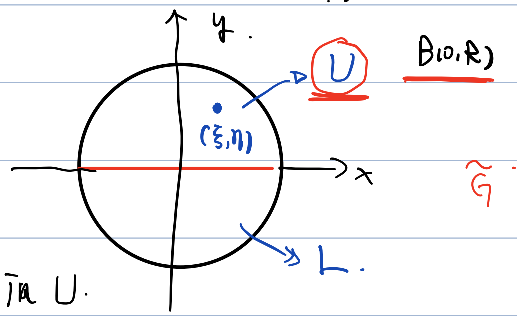
②. 半平面 $H^n = \{x\in\mathbb{R}^n : x_n > 0\}$.

$G(x;\xi) = \Gamma(x,\xi) - \Phi(x-\hat{\xi})$

$= \Phi(x-\xi) - \Phi(x-\hat{\xi}).$

其中 若 $\xi=(\xi_1,\cdots\xi_n)$ 则 $\tilde{\xi}=(\xi_1,\cdots\xi_{n-1},-\xi_n)$

③. 半球



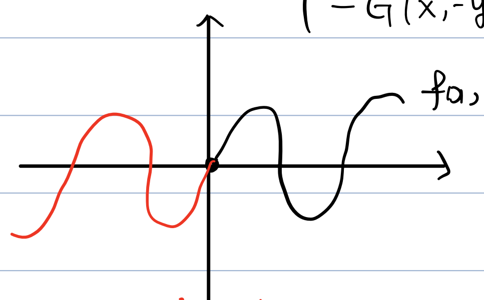
求 U 上的 Green function

$$G(x,y;\xi,\eta)$$

则 $\begin{cases} -\Delta G = \delta(x-\xi,\ y-\eta) & \text{in } U \\ G|_{\partial U} = 0. \end{cases}$

奇延拓.

$$\tilde{G}(x,y;\xi,\eta)=\begin{cases} G(x,y;\xi,\eta) & U \quad (y>0) \\ -G(x,-y;\xi,\eta) & L. \quad (y<0). \end{cases}$$



奇延拓 保证 偏导连续 y=0 处.

$$\frac{\partial\tilde{G}}{\partial y}(x,0;\xi,\eta)=\begin{cases} \lim\limits_{y\to0^+}\dfrac{G(x,y;\xi,\eta)}{y}. \\ \lim\limits_{y\to0^-}\dfrac{-G(x,-y;\xi,\eta)}{y} \overset{-y\mapsto y}{=} \lim\limits_{y\to0^+}\dfrac{G(x,y;\xi,\eta)}{y}. \end{cases}$$

$\Rightarrow \dfrac{\partial\tilde{G}}{\partial y}$ 连续.

在ξ,η处∞ 其余为0.

$$-\Delta\tilde{G} = -\Delta G(x,y;\xi,\eta) = \delta(x-\xi,\ y-\eta) \quad \text{in } U$$

$$-\Delta\tilde{G} = -\Delta(-G(x,-y;\xi,\eta)) = \Delta G(x,-y;\xi,\eta)$$

$$= -\delta(x-\xi,\ -y+\eta) \quad \text{in } L.$$

(ξ, -η) +∞, 其余为0.

$$\Rightarrow \begin{cases} -\Delta\tilde{G}(x,y;\xi,\eta) = \delta(x-\xi,\ y-\eta) - \delta(x-\xi,\ -y-\eta). \\ \tilde{G}|_{\partial B(0,R)} = 0 \end{cases}$$

$$\boxed{\begin{cases} -\Delta G_1(x,y;\xi,\eta) = \delta(x-\xi, y-\eta) \\ G_1|_{\partial B(0,R)} = 0 \end{cases}} \qquad \boxed{\begin{cases} -\Delta G_2(x,y;\xi,\eta) = -\delta(x-\xi, -y-\eta) \\ G_2|_{\partial B(0,R)} = 0. \end{cases}}$$

$$\tilde{G} = G_1 + G_2 = \left(\qquad\qquad \right).$$

$$G = \tilde{G}|_U$$

④ 第一象限.

$\hat{\xi} = (\xi_1, -\xi_2)$.

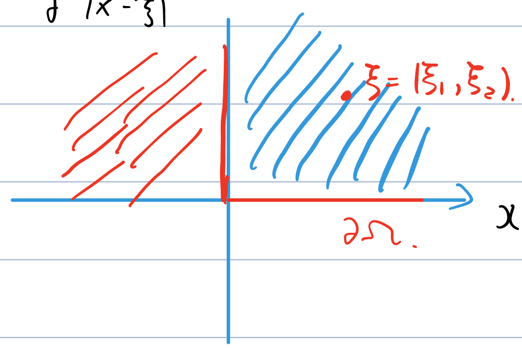
已知：上半平面 Green function $\quad x=(x_1, x_2)$, $\xi = (\xi_1, \xi_2)$

$$G_0(x;\xi) = \Phi(x-\xi) - \Phi(x-\hat{\xi})$$

$$\boxed{\Phi(x) = \frac{1}{2\pi}\log\frac{1}{|x|}}$$

$$= \frac{1}{2\pi}\log\frac{1}{|x-\xi|} - \frac{1}{2\pi}\log\frac{1}{|x-\hat{\xi}|}$$

第一象限 Green function:



$G(x;\xi)$, 满足

$$\begin{cases} -\Delta G = \delta(x-\xi). \\ G|_{\partial\Omega} = 0. \end{cases}$$

$$\tilde{G}(x;\xi) = \begin{cases} G(x;\xi) & x_1 \geq 0 \\ -G(-x_1, x_2; \xi_1, \xi_2) & x_1 < 0 \end{cases}$$

$$\tilde{G}(x;\xi) = \delta(x-\xi) - \delta(-x_1-\xi_1, x_2-\xi_1).$$

$$= \frac{1}{2\pi}\log\frac{1}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} - \frac{1}{2\pi}\log\frac{1}{\sqrt{(x_1-\xi_1)^2+(x_2+\xi_2)^2}}$$

$$-\left(\frac{1}{2\pi}\log\frac{1}{\sqrt{(x_1+\xi_1)^2+(x_2-\xi_2)^2}} - \frac{1}{2\pi}\log\frac{1}{\sqrt{(x_1+\xi_1)^2+(x_2+\xi_2)^2}}\right).$$

$$G(x;\xi) = \tilde{G}(x;\xi)\Big|_{\substack{x>0\\ y>0}}$$

$$u(\xi) = -\int_\Omega G(x;\xi)\,\Delta u\,dx - \int_{\partial\Omega} \frac{\partial G}{\partial \vec{n}}(x;\xi)\, u\, dS.$$

$$\begin{cases} -\Delta u = f & \Omega \\ u = \varphi & \partial\Omega. \end{cases}$$

8. $$\begin{cases} -\Delta u = 0 & \boxed{x^2+y^2<R} \\ u = A + B\sin\theta & x^2+y^2 = R^2. \end{cases}$$

$$(x,y) = (r\cos\theta, r\sin\theta)$$

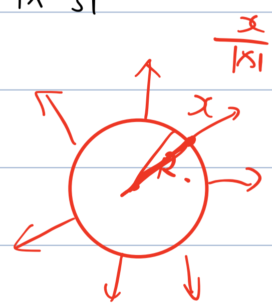
求 $u(x,y) = ?$

Solution: $$G(x;\xi) = \Phi(x-\xi) - \Phi\left(\frac{|\xi|}{R}(x-\tilde{\xi})\right).$$

$$= \frac{1}{2\pi}\log\frac{1}{|x-\xi|} - \frac{1}{2\pi}\log\frac{R}{|\xi|\cdot|x-\tilde{\xi}|}.$$

$$\tilde{\xi} = \frac{R^2}{|\xi|^2}\xi.$$

$$u(\xi) = -\int_{\partial\Omega}\frac{\partial G}{\partial \vec{n}}(x;\xi) f\, dS.$$



求 $$\frac{\partial G}{\partial \vec{n}} = \nabla G\cdot\vec{n}$$

$$= \nabla G \cdot \frac{x}{|x|}$$

$$\frac{\partial G}{\partial x_i}(x;\xi) = \frac{-1}{2\pi}\frac{\partial}{\partial x_i}\left(\log|x-\xi| - \log|x-\tilde{\xi}|\right).$$

$$= \frac{-1}{2\pi}\left(\frac{1}{|x-\xi|}\frac{\partial|x-\xi|}{\partial x_i} - \frac{1}{|x-\tilde{\xi}|}\frac{\partial|x-\tilde{\xi}|}{\partial x_i}\right)$$

$$= \frac{-1}{2\pi}\left(\frac{x_i-\xi_i}{|x-\xi|^2} - \frac{x_i - \frac{R^2\xi_i}{|\xi|^2}}{|x-\tilde{\xi}|^2}\right).$$

$$\frac{|\xi|^2 x_i - R^2\xi_i}{\left|\,|\xi|x - \frac{R^2\xi}{|\xi|}\right|^2}$$

$$\nabla G\cdot\vec{n} = \frac{1}{R}\frac{1}{2\pi}\left(\frac{\sum x_i^2 - x_i\xi_i}{|x-\xi|^2} - \frac{\sum|\xi|^2 x_i^2 - R^2\xi_i x_i}{\left|\,|\xi|x - \frac{R^2\xi}{|\xi|}\right|^2}\right).$$

$$= \frac{1}{2\pi R}\left(\frac{|x|^2 - x\cdot\xi}{|x-\xi|^2} - \frac{|\xi|^2|x|^2 - R^2 x\cdot\xi}{\left|\,|\xi|\,x - \frac{R^2\xi}{|\xi|}\right|^2}\right).$$

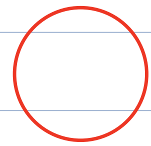
$x\in\partial B(0,R)$ 时 $|x| = R$

$n=2$ 时 (P180).

$$\frac{\partial G}{\partial \vec{n}} = \frac{1}{2\pi R}\,\frac{r^2-R^2}{r^2+R^2-2rR\cos(\theta-\alpha)}.$$

$$(x_1, x_2) = (R\cos\alpha,\ R\sin\alpha)$$

$$(\xi_1, \xi_2) = (r\cos\theta,\ r\sin\theta).$$



$$u(\xi) = -\int_{\partial\Omega}\frac{\partial G}{\partial \vec{n}} f\, dS(x).$$

$x_1 = R\cos\alpha$

$x_2 = R\sin\alpha.$

$$= -\int_0^{2\pi}\frac{\partial G}{\partial \vec{n}} f\ R\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-R^2}{r^2+R^2-2rR\cos(\theta,\alpha)} f(\theta)\, d\theta.$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-R^2}{r^2+R^2-2rR\cos(\theta-\alpha)}\cdot(A+B\sin\theta)\, d\theta.$$

一. 1. 第一章.

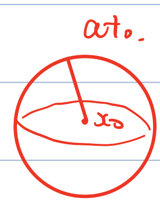
2、依赖区间 (x_0, t_0)

$$\begin{cases} u_{tt}-a^2u_{xx}=f \\ u|_{t=0}=\varphi \\ u_t|_{t=0}=\psi \end{cases}$$

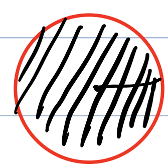
$(x_0-at_0,\ x_0+at_0)$

$$u(x,t) = \frac{1}{4\pi a^2 t}\int_{\partial B(x,at)}\psi(y)\, dS.$$

x_0 t_0. $\partial B(x_0, at_0)$



$$u(x,t) = \frac{1}{2\pi a}\int_{B(x,at)}\frac{\psi(y)}{\sqrt{}}\, dy.$$



3. P53.

P. 定义 P186

10. u_{xx} , u_{xy} u_{yy}

x^2 xy y^2.

$a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}=0.$

$a_{11}x^2+2a_{12}xy+a_{22}y^2$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}. = A(x,y)$

椭圆 → A的特征值均正/负

双曲 → 一正一负

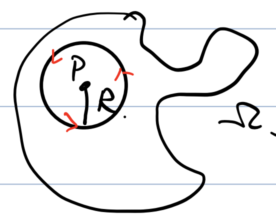
抛物 → 有一个是0.

$xU_{xx}+y\,U_{yy} \longrightarrow \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}. \longrightarrow xy$

⟹ 在 $\{(x,y): xy<0\}$ 上是双曲.

4 调和函数 (193).

$-\Delta u=0$ Ω



① u 解析 (无穷次可导)

② 平均值原理.

u 在 Ω 内调和, , 则 $\forall\, P=(x_0,y_0)\in\Omega$ 且 $\overline{B(P_0,R)}\subset\Omega$.

则. $u(x_0,y_0)=\frac{1}{2\pi R}\int_{\partial B(P_0,R)} u\,dl.$

$\partial B(P_0,R)$的周长.

③ Liouville.

u 在 $\mathbb{R}^2$ 上有界, 则 u 为常数

$u(1,\theta) = \sin^2\theta$. u 在 $\partial B(0,1)$ 上的取值.

$$u(0,0) = \frac{1}{2\pi R}\int_{\partial B(0,1)} u(x)\, dS$$

$$= \frac{1}{2\pi R}\int_0^{2\pi} u(1,\theta)\, R\, ds.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin^2\theta\, d\theta.$$

$$= \frac{4}{2\pi}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta.$$

$$= \frac{4}{2\pi}\frac{1!!}{2!!}\frac{\pi}{2} = 1$$

瓦里斯公式

$$\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \begin{cases} \frac{(n-1)!!}{n!!} & n\text{奇} \\ \frac{(n-1)!!}{n!!}\frac{\pi}{2} & n\text{偶} \end{cases}$$

三.2. 参考 P213. 6.

$$\begin{cases} -\Delta u^3 + u^3 \overset{?}{=} 0 & \Omega. \\ \frac{\partial u}{\partial n} + \alpha(x) u|_{\partial\Omega} = \varphi & \alpha(x) \geq \alpha_0 > 0. \end{cases}$$

有最大模估计

$$\max_{\Omega} |u(x)| \leq \frac{1}{\alpha_0}\max_{\partial\Omega} |\varphi(x)|.$$

Laplace 弱极值原理. P185.

$Lu = -\Delta u + c(x)u = f$. $c(x) \geq 0$ in Ω.

P185.(2.2). $c(x) \geq 0$ 且在 Ω 上有界, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ 且.

$Lu = f \leq 0$. $f \mapsto -f$.

则. $\sup_{x\in\Omega} u(x) \leq \sup_{\partial\Omega} u^+(x)$

$u^+ = \max(u, 0)$.

用法:

$$\begin{cases} Lu = f \geq 0. \\ u|_{\partial\Omega} \geq 0. \end{cases}$$

则 $u|_{\Omega} \geq 0$.

证明 $\begin{cases} -\Delta u + u^7 = 0 & B(0,1). \\ u|_{|x|=1} = 0 \end{cases}$ 只有零解.

Proof: $\begin{cases} -\Delta u + u^7 = 0 & B(0,1). \\ u|_{|x|=1} = \varphi \end{cases}$ (*)

u 满足(*)，则 u 满足

$\begin{cases} -\Delta v + u^6 v = 0 \\ v|_{|x|=1} = \varphi \end{cases}$ $u^6 \to C(\bar{\Omega})$. $\Omega = B(0,1)$.

构造 $w = \Phi \pm u$. $\Phi = \sup_{\partial\Omega} |\varphi|$

直接验证: $\begin{cases} \cdot\ -\Delta w + u^6 w \geq 0 \\ \cdot\ w|_{\partial\Omega} = \Phi \pm u|_{\partial\Omega} = \Phi \pm \varphi \geq 0. \end{cases}$

$\Rightarrow \quad w \geq 0 \quad \text{in } \Omega$

即. $\Phi \pm u \geq 0$

$\Rightarrow \quad |u| \leq \Phi.$

取 $\Phi = 0$, 则 $|u| = 0 \Rightarrow$ u 只有零解.